Comprehensive Exam in Mathematics

Morning Part

Instructions:

No calculators or reference material can be used.

Justify your answers as much as possible, but briefly!

For an MA pass, attempt (solution need not be perfect; partial credit is awarded e.g. for indicating methods and giving examples) at least: (1); (2); (3); (4); (5); (6); (8); (9); one of (10-11); (12); one of (13-14). For a PhD pass, same as before plus (7) but minus (8). If you have time to do more than one out of a group, you get more credit.

1. Check the following series for convergence or divergence:

   (a) $\sum_{k=1}^{\infty} k^p p^k$, $p > 0$
   
   (b) $\sum_{k=1}^{\infty} (k\sqrt{k} - 1)^k$
   
   (c) $\sum_{k=1}^{\infty} k!/k^k$
   
   (d) $\sum_{k=2}^{\infty} 1/(\log k)^p$, $p > 0$

2. Prove the parallelogram law: $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$ for all $x, y \in \mathbb{R}^n$.

3. Let $\xi = (\xi_1, \xi_2)$, $e_1 = (1, 0)$, $e_2 = (0, 1)$ and $x, x_1, x_2 \in \mathbb{R}^n$. Find the following limits, if they exist (or show that they don’t exist):

   (a) $\lim_{\xi \to 0} \frac{\xi_1^4 + \xi_2^4}{\xi_1^4 + \xi_2^4}$
   
   (b) $\lim_{\xi \to 0} \frac{\xi_1 \xi_2^2}{\xi_1^2 + \xi_2^4}$
   
   (c) $\lim_{\xi \to \xi_0} \frac{\xi_1^2 \xi_2^2}{(\xi_1^2 \xi_2^2 + (\xi_1 - \xi_2)^2)}$
   
   (d) $\lim_{\xi \to \xi_0} \xi_1 \xi_2 / (\xi_1^2 + \xi_2^2)$, $\xi_0 = e_1 + e_2$
   
   (e) $\lim_{\|x\| \to \infty} \frac{\|x - x_1\|}{\|x - x_2\|}$
   
   (f) $\lim_{\|x\| \to \infty} \frac{(x \cdot x_1)(x \cdot x_2)}{x \cdot x}$

4. Find a general solution to the differential equation $y'' - 2y' + y = x^{-1}e^x$.

5. A spring for which $k = 48$ lb/ft has a 16-lb mass attached;
   
   (a) Find the natural frequency and period of the motion.
   
   (b) If the mass is released from rest 2 in. below its equilibrium position, find the subsequent motion. Neglect damping.
(6) Let 

\[ f_k(x) = \begin{cases} 
  x^k \sin(1/x), & x \neq 0, \\
  0, & x = 0, 
\end{cases} \]

\( k \in \mathbb{N} \). Discuss the differentiability/differentiation of \( f_k \) (in the variable \( x \)).

(7) Let \( f : (0, 1) \to \mathbb{R} \) be differentiable and \( |f'(x)| \leq 1 \) for all \( x \). If \( x_k = f(1/k) \), show that \( \{x_k\} \) converges.

(8) Write the equation of the tangent plane to the surface \( x^2 + y^2 + z^2 = c \) at any point \((x_0, y_0, z_0)\) on the surface.

(9) Examine each of the following series for uniform convergence:

(a) \( \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2} \)

(b) \( \sum_{k=1}^{\infty} \frac{1}{x - 1} \)

(c) \( \sum_{k=1}^{\infty} \frac{1}{(x - k)^2} \)

(10) Determine which of the following integrals exist:

(a) \( \int_0^1 \frac{\sqrt{x}}{\ln x} \, dx \)

(b) \( \int_0^1 \frac{\sin x}{x^{3/2}} \, dx \)

(c) \( \int_0^2 \frac{x^2 + 1}{x^2 - 4} \, dx \)

(11) Determine which of the following integrals exist:

(a) \( \int_1^\infty e^{-x} \ln x \, dx \)

(b) \( \int_3^\infty \frac{x^2 + 1}{x^2 - 4} \, dx \)

(c) \( \int_1^\infty \frac{\sin(1/x)}{x} \, dx \)

(12) Assume a particle of charge \( q \) is located at the origin of \( x, y, z \) space. Then its electrical potential function is given by

\[ \phi(x) = \phi(x, y, z) = \frac{q}{|x|} = \frac{q}{\sqrt{x^2 + y^2 + z^2}}, \]

where \( x = (x, y, z) \). Let \( \mathbf{F}(x, y, z) \) be defined as the force that a particle of charge 1 would feel at position \((x, y, z)\), due to the particle at the origin. Then

\[ \mathbf{F}(x) = -\nabla \phi(x). \]

The flux of \( \mathbf{F} \) (the force field) across a smooth surface \( S \) is defined to be the surface integral of \( \mathbf{F} \) over \( S \). Show that the flux of \( \mathbf{F} \) across a sphere of radius \( a \) center at the origin is independent of \( a \). What does it equal? Does that make sense to you in terms of how you would expect the electrical force field of a particle to be related to its charge?
(13) Can the system
\[
\begin{align*}
y_1 &= x_1 + x_1x_2x_3, \\
y_2 &= x_2 + x_1x_2, \\
y_3 &= x_3 + 2x_1 + 3x_3^2,
\end{align*}
\]
be solved for \( \mathbf{x} \) in terms of \( \mathbf{y} \) near \((0, 0, 0)\)?

(14) Can the equation \( xy - z \ln y + e^{x^2} - 1 = 0 \) be solved for \( z \) in terms of \( x \) and \( y \) near \((0, 1, 1)\)? How about for \( y \) in terms of \( x \) and \( z \)?