Torelli Theorem of $ALH^*$ Gravitational Instantons

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Outline of the Talk

- Gravitational Instantons and their Classifications
- $ALH^*$ Gravitational Instantons
- From SYZ Conjecture to Gravitational Instantons
- Torelli Theorem of $ALH^*$ Gravitational Instantons
- Compactification Result of $ALH^*$ Gravitational Instantons
Gravitationaional Instantons

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- Mathematically, gravitational instantons are non-compact complete hyperKähler 4-manifolds with $L^2$ curvature.
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- Mathematically, gravitational instantons are non-compact complete hyperKähler 4-manifolds with $L^2$ curvature.
- A 4-manifold $X$ is hyperKähler if it admits
  - a Kähler form $\omega$
  - a holomorphic volume form $\Omega$
  s.t. the complex Monge-Ampere equation $2\omega^2 = \Omega \wedge \bar{\Omega}$ holds.
- They arise as the blow up limits of hyperKähler metrics.
  Ref: Foscolo’16, HSVZ ’18, SZ’19, CVZ ’19.
Classification of Gravitational Instantons
From the volume growth, people found gravitational instantons of type $ALE(r^4), ALF(r^3), ALG(r^2), ALH(r)$. 

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(Hein ’12) constructed two new gravitational instantons $ALG^*, ALH^*$ with volume growth $r^2, r^{4/3}$. The former has a different curvature decay from $ALG$.

(Sun-Zhang ’21) The above are the exhaustive list of gravitational instantons.
Rational elliptic surfaces (RES) are projective rational surface with an elliptic fibration structure. They are always realized as blow up of the base points (9 point, possibly infinitely near) of a cubic pencil in $\mathbb{P}^2$. 
Rational Elliptic Surfaces

- **Rational elliptic surfaces** (RES) are projective rational surface with an elliptic fibration structure.
- They are always realized as blow up of the base points (9 point, possibly infinitely near) of a cubic pencil in $\mathbb{P}^2$.
- (Hein ’12) $ALH, ALG, ALG^*, ALH^*$ gravitational instantons can be realized on the complement of a fibre in a rational elliptic surface, with the fibre is of type $l_0$, finite monodromy, $l_k^*$ or $l_k$. 
(Kroheimer ’89) Torelli theorem for ALE and period domain. Two ALE gravitational instantons with the HK triples in the same cohomology class, the HK triples are the same.

(Chen-Chen ’15 ’16) ALF, ALH case.
(Kroheimer ’89) Torelli theorem for $ALE$ and period domain. Two $ALE$ gravitational instantons with the HK triples in the same cohomology class, the HK triples are the same.

(Chen-Chen ’15 ’16) $ALF$, $ALH$ case.

**Theorem (Collins-Jacob-L’21)**

$(X_i, \omega_i, \Omega_i)$ $ALH^*$ gravitational instantons. If $f : X_2 \to X_1$ diffeo. s.t. $f^* [\omega_1] = [\omega_2], f^* [\Omega_1] = [\Omega_2]$. Then $\exists F : X_2 \to X_1$ s.t. $F^* \omega_1 = \omega_2, F^* \Omega_1 = \Omega_2$.

(Chen-Viaclovsky-Zhang ’21) $ALG$, $ALG^*$ case.
Calabi-Yau Manifolds and SYZ Conjecture
Calabi-Yau Manifolds

- **Calabi-Yau $n$-fold $X$**
  - higher dimension analogue of elliptic curves.
  - complex manifold $X$ with
    1. nowhere vanishing holomorphic $n$-form $\Omega$
    2. $d$-closed non-degenerate positive $(1, 1)$-form $\omega$ such that $\omega^n = c\Omega \wedge \bar{\Omega}$.

Examples:
- degree 5 hypersurface in $\mathbb{P}^4$ (quintic 3-fold).
- (Tian-Yau) complement of a smooth anti-canonical divisor in a Fano manifold
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Conjecture (Strominger-Yau-Zaslow '96)

- Calabi-Yau manifolds admit special Lagrangian torus fibration near large complex structure limit.

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1. (Harvey-Lawson ’82) A submanifold $L$ in $X$ is special Lagrangian if $\omega|_L = 0$, $\text{Im}\Omega|_L = 0$.

2. (Duistermaat ’80) Any compact fibre of a Lagrangian fibration is topologically a torus.
Why SYZ Conjecture?

The SYZ conjecture is important in various aspects:

- It gives a geometric description of Calabi-Yau manifolds.
- It provides a recipe to construct the mirror $\hat{X}$.
- (Leung-Yau-Zaslow, FLTZ, CPU, FHLY, CHL,...) It gives a guidance of how branes mirror to each other in the homological mirror symmetry.
- It motivates the method of family Floer mirror and Gross-Siebert program in mirror symmetry.
\textit{ALH}^* Gravitational Instantons
Calabi Model and $ALH^*$ Gravitational Instantons

- $D$ elliptic curve, $L$ line bundle on $D$ w/ $\deg L = k > 0$.
- $Y_C =$ neighborhood of zero section of $L$, $\pi : Y_C \to D$.
- $X_C = Y_C \setminus D$.
- $\Omega_C = \frac{dw}{w} \land \pi^* \Omega_D$.
- $\omega_C = \sqrt{-1} \frac{2}{3} \partial \bar{\partial} (- \log |\xi|^2_h)^{\frac{3}{2}}$, where $\omega_D = \sqrt{-1} \partial \bar{\partial} h$.
- Set $l_0 = \left( - \log |\xi|^2_h \right)^{\frac{1}{4}}$. Then
  1. $|\nabla^k Rm| \leq C_k l_0^{-(k+2)}$ has good control and
  2. $C_t^{-1} l_0^{-1} \leq \text{inj} \leq C_t l_0^{-1}$ degenerates.
- (Sun-Zhang ’21) $ALH^*$ gravitational instantons are exponentially decay to the Calabi model at infinity.
Examples of $ALH^*$ Gravitational Instantons

- (Tian-Yau '90) $Y$ weak del Pezzo surface, $D$ smooth anti-canonical, then $X = Y \setminus D$ is $ALH^*$.

- (Hein '12) $\tilde{Y}$ rational elliptic surface, $\tilde{D}$ $I_k$-fibre, then $\tilde{X} = \tilde{Y} \setminus \tilde{D}$ is $ALH^*$. 

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- (Hein ’12) $\check{Y}$ rational elliptic surface, $\check{D}$ $I_k$-fibre, then $\check{X} = \check{Y} \setminus \check{D}$ is $ALH^*$.
- (Hein ’12) They have the same asymptotics of $inj$, $Rm$, volume growth,... are all the same.
- Are these two examples related by HK rotation?
The hyperKähler triple \((\omega, \Omega)\) induces an \(S^2\)-family of complex structures on the underlying space of \(X\).

Then holomorphic curves in \(X\) \(\Leftrightarrow\) special Lagrangians in \(X_{\theta}\).
Relations between Two $ALH^*$ Gravitational Instantons

Q: Are two $ALH^*$ gravitational instantons related by HK rotation?

Theorem (CJL '19)
Suitable HK rotation $\tilde{X}$ of $X$ can be compactified to a rational elliptic surface $\tilde{Y}$.

Q: Is the metric on $\tilde{X}$ the one constructed by Hein?

Theorem (CJL '20)
There exists a $\mathbb{R}$-family of HK metrics on $\tilde{X}$ with $Z \subseteq \mathbb{R}$ corresponds Hein's metric.

Discovery of "non-standard semi-flat metric", new ansatz for HK metrics.
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Uniformization of $ALH^*$ Gravitational Instantons

**Theorem (CJL’ 21, HSVZ ’21)**

*Up to HK rotation, any $ALH^*$ gravitational instanton $X$ can be compactified to a rational elliptic surface.*

- Special Lagrangian tori fibration in $X_C$.
- Lagrangian mean curvature flow the ansatz fibration to a special Lagrangian near infinity of $X$.
- $\tilde{X}$ HK rotation of $X$ can be compactified to a compact complex surface $\tilde{Y}$.
- Enrique-Kodaira classification $\Rightarrow \tilde{Y}$ is RES.

(Persson) Classification of singular fibres in RES $\Rightarrow 1 \leq k \leq 9$.

10 different diffeomorphism types of $ALH^*$ gravitational instantons.
Lagrangian Mean Curvature Flow

Let $L$ be a graded Lagrangian submanifold in $X$, i.e.,

\[ \exists \text{ the phase } \theta : L \to \mathbb{R} \text{ is the function such that} \]

\[ \Omega|_L = e^{i\theta} \text{vol}_L. \]

$L$ is a special Lagrangian if $\theta$ is a constant.
Lagrangian Mean Curvature Flow

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  $$\Omega|_L = e^{i\theta} \text{vol}_L.$$  

  $L$ is a special Lagrangian if $\theta$ is a constant.

- The mean curvature $\vec{H} = J \nabla \theta$ and the mean curvature flow is
given by evolving family of immersions $F_t : L \to X$ with

  $$\frac{\partial}{\partial t} F_t = \vec{H}.$$  

- (Smoczyk) Maslov zero Lagrangian condition is preserved
  under mean curvature flow in Kähler–Einstein manifolds.

- Smooth Convergent Limit of LMCF gives Special Lagrangians.
The Torelli Theorem
K3 surfaces are simply connected, compact surface with trivial canonical bundles.

\[ \mathbb{L}_{K3} := H^2(K3) \cong U^{\oplus 3} \oplus E_8^{\oplus 2}, \text{ where } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

\([\Omega] \in \mathbb{P}(\mathbb{L}_{K3} \otimes \mathbb{C})\) determines the complex structure if 
\([\Omega] \wedge [\Omega] = 0, [\Omega] \wedge [\bar{\Omega}] > 0.\]

(Yau ’76) Unique Ricci-flat metric in each Kähler class.

We will follow the same idea to prove the Torelli theorem of \(ALH^*\) gravitational instantons.
Theorem (Gross-Hacking-Keel ’15)

\((\tilde{Y}_i, \tilde{D})\) log Calabi-Yau surfaces w/ isom. \(\mu : \text{Pic}(\tilde{Y}_1) \to \text{Pic}(\tilde{Y}_2)\),

1. \(\mu\) preserves the periods \(\phi_i \in \text{Hom}(\tilde{D}^\perp, \text{Pic}^0(\tilde{D}))\),

\[\phi_i : L \mapsto L|_{\tilde{D}}.\]

2. \(\mu\) preserves \((-2)\)-curve classes

3. \(\mu\) sends one Kähler class to a Kähler class.

Then \(\mu = F^*\), w/ isomorphism of pairs \(F : (\tilde{Y}_2, \tilde{D}) \to (\tilde{Y}_1, \tilde{D}).\)

Without the 3rd assumption, \(F^*\) may be differed by an element in the Weyl group.
Proof of Torelli theorem for ALH* Gravitational Instantons

- Construct an isometry of lattices \( \text{Pic}(\tilde{Y}_1) \to \text{Pic}(\tilde{Y}_2) \) from \( f^* \).
  - Suitable homotopic modification of \( f \) to preserve sections.
  - Computation of \( \text{MCG}(X_C) \).
- GHK’15 \( \Rightarrow \exists F : (\tilde{Y}_2, \tilde{D}) \cong (\tilde{Y}_1, \tilde{D}) \).
  - (Looijenga) \( \phi_i = \exp(2\pi i \int \tilde{\Omega}) \).
  - Discrepancy of Kähler classes on \( \tilde{X} \) and \( \tilde{Y} \).

**Theorem (CJL ‘20)**

HK metrics on \( \tilde{X} \) with non-standard semi-flat metrics asymptotics and the same cohomology class are differed by translation of sections.

Replace \( F \) by composition w/ translation of sections.
It still remains to ask given cohomology class of $[\omega], [\Omega]$, is there a gravitational instantons to realize the triple for the case of $ALG, ALG^*, ALH^*$?

Obvious obstruction is $[\omega], [\Omega]$ can not simultaneously vanish on $(-2)$-classes.

Chen-Viaclovsky-Zhang ’21 conjectured no other obstruction.
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**Theorem (Lee-Lin, work in progress)**

*Period domain for $ALG, ALG^*, ALH^*$ gravitational instantons.*
Applications
Recall that given a primitive element in $H_1(D, \mathbb{Z})$, there is a corresponding SYZ fibration in $X = Y \setminus D$.

(Hacking-Keating ’21) study the symplectomorphism of Looijenga pairs. Conjecture that the monodromy of the moduli of pairs $(Y, D)$ sends Lagrangian fibration to Lagrangian fibration.

**Theorem (Lau-Lee-L.- ’22)**

*Given a monodromy action on $H_2(X, \mathbb{Z})$, there exists an isometry of $X$ realizing the monodromy and sending one SYZ fibration to another.*

Maybe there is some application of lattice theory on this geometry?
Theorem (Collins-Jacob-L.- '21)

Let $Y$ be a weak del Pezzo surface, $D$ be a smooth anti-canonical divisor and $X = Y \setminus D$. Fix a meromorphic 2-form $\Omega$ on $Y$ with a simple pole along $D$. For each Kähler class of $X$, there exists a unique hyperKähler metric $\omega$ in the given Kähler class with $L^2$ curvature.

This partially answers a question of Tian-Yau but can the $L^2$-condition be removed?
Compactification to Weak Del Pezzo Surfaces

Consider the HK rotation map

$$\Psi : \mathcal{M}_k \to \mathcal{M}_k$$

$$((Y, D), \mu, c, [\omega], \gamma) \mapsto ((\tilde{Y}, \tilde{D}), \tilde{\mu}, [\tilde{\omega}], \alpha)$$

- $\text{Im}\Psi$ is open.
- $\text{Im}\Psi$ is closed.
  - Construct log CY pairs with perturbed periods.
  - $\Psi$ continuous, some structure of moduli space of log CY pairs.
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- $\text{Im} \Psi$ is open.
  - construct log CY pairs with perturbed periods.
- $\text{Im} \Psi$ is closed.
  - $\Psi$ continuous, some structure of moduli space of log CY pairs.
- $\Psi$ is surjective and Torelli theorem $\Rightarrow$ $X$ compactified to $Y$.

**Theorem (HSVZ '21, CL'22)**

*Every ALH* gravitational instanton (up to HK rotation) can be compactified to a weak del Pezzo surface.*
The compactification result further motivates the question:

**Question**

*Can ALG, ALG*-gravitational instantons compactified to algebraic surfaces other than rational elliptic surfaces after suitable HK rotations?*

**Theorem (Collins-L.-, in progress)**

*True for gravitational instantons of second Betti number 5.*

This has further applications in mirror symmetry: for instance how Gross-Hacking-Keel mirror symmetry is compatible with SYZ mirror symmetry, computation of certain local open GW invariants, how SYZ fibration detects the superpotentials...etc.
THANK YOU!