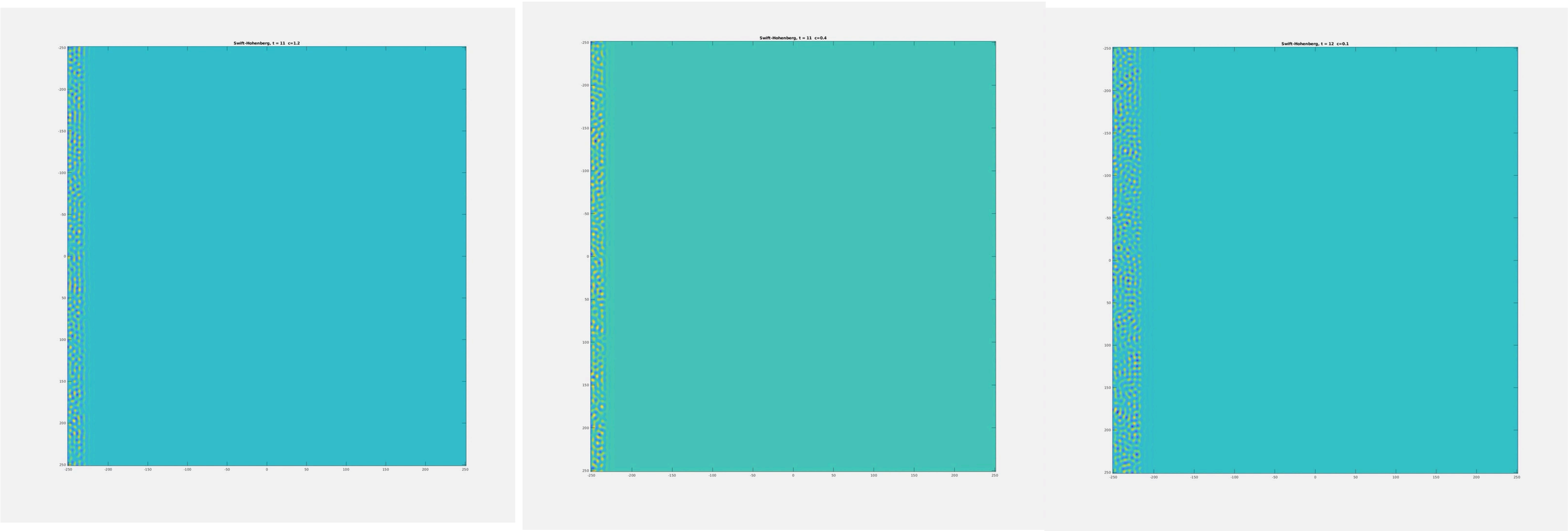
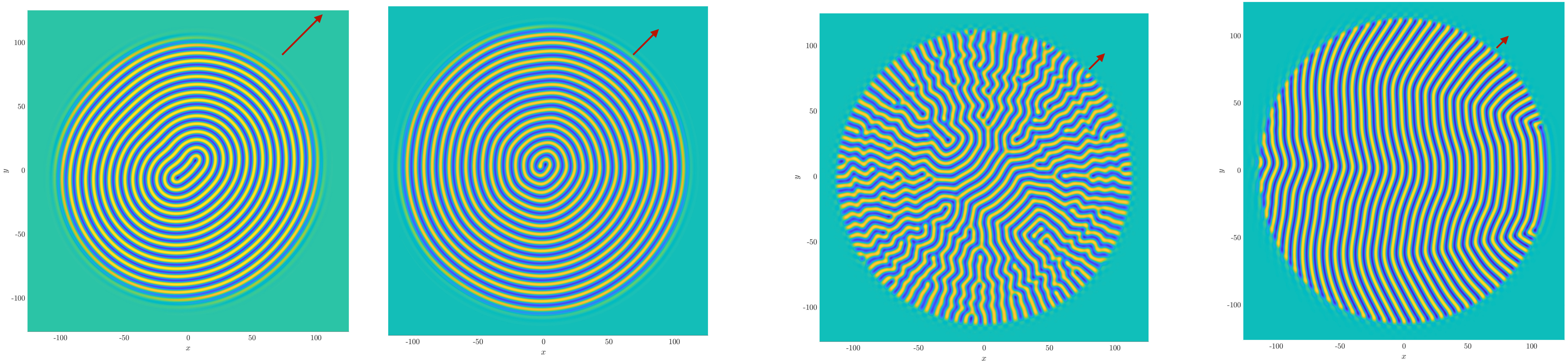
Fronts in the wake of a parameter ramp:

slow passage through folds and pitchforks

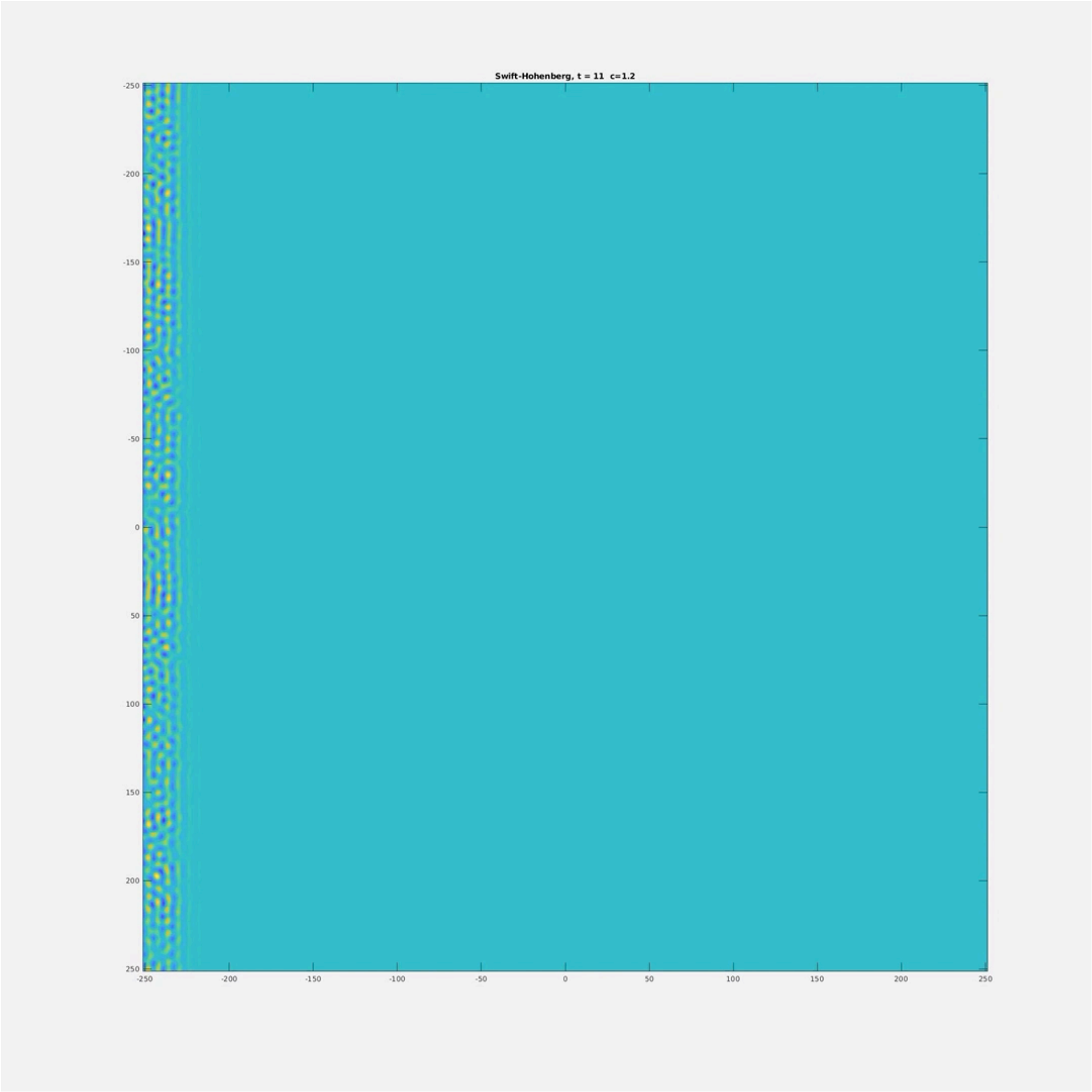
BKT 2024
Ryan Goh

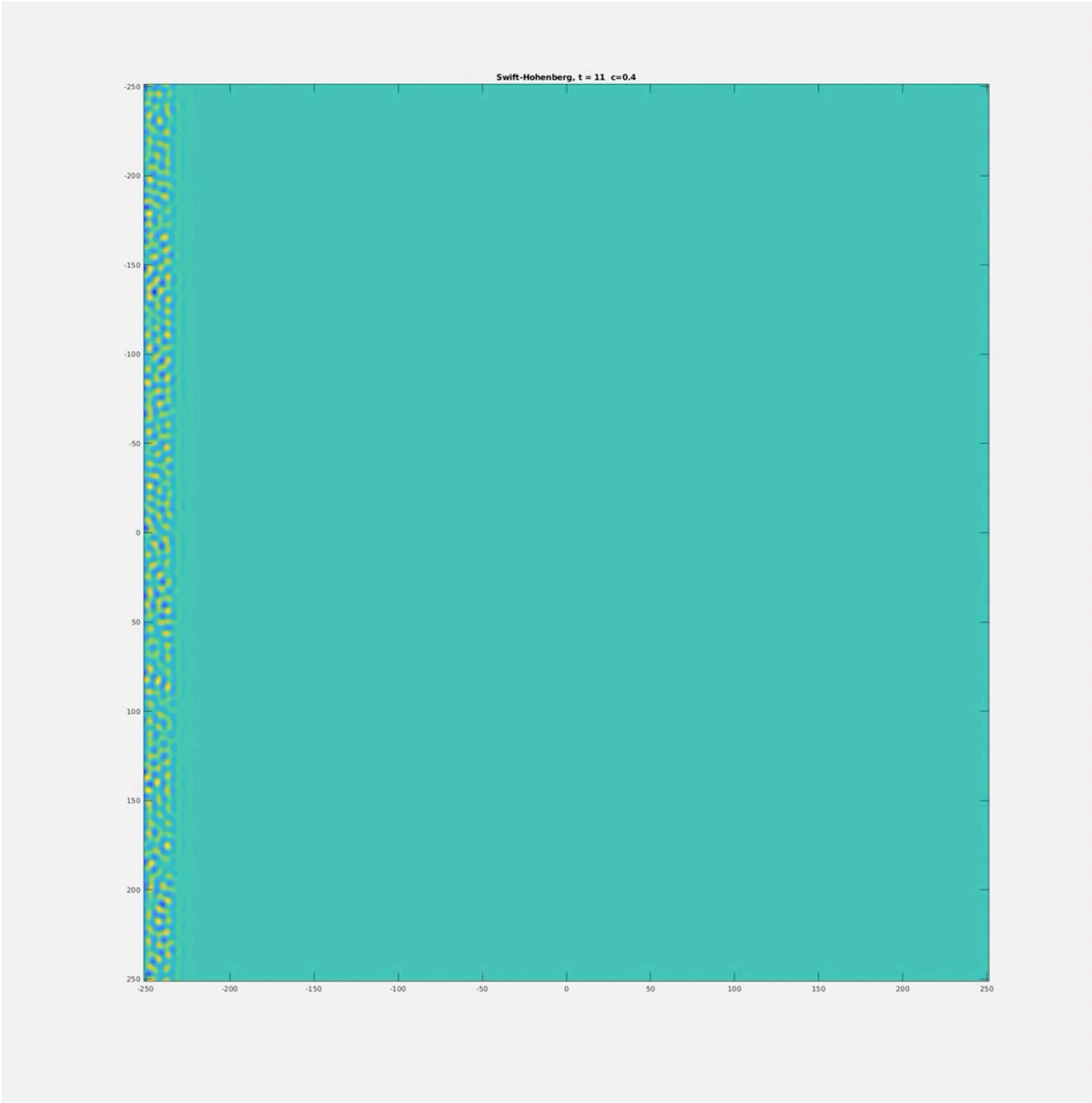
How does growth and heterogeneity control pattern formation?

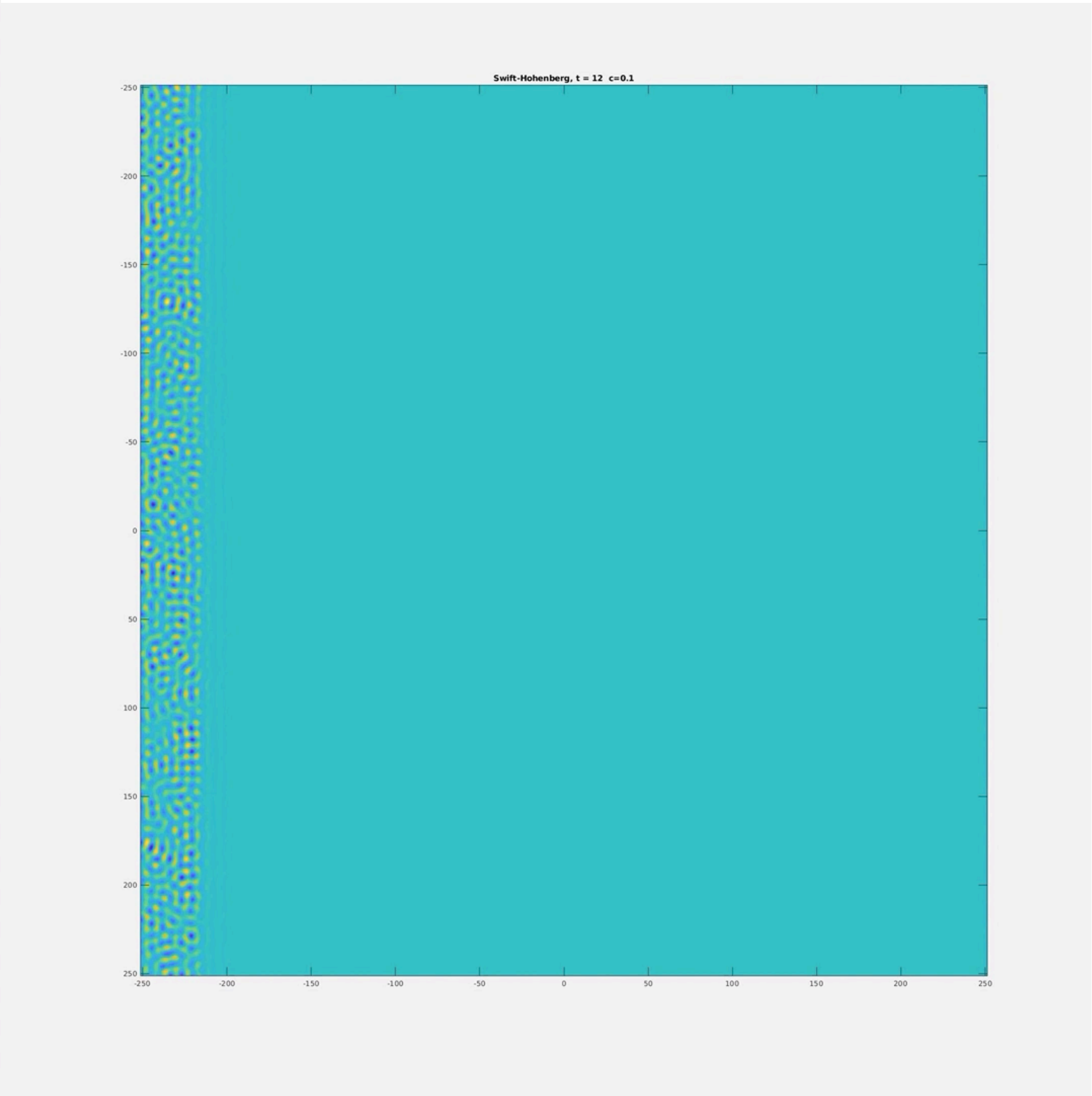


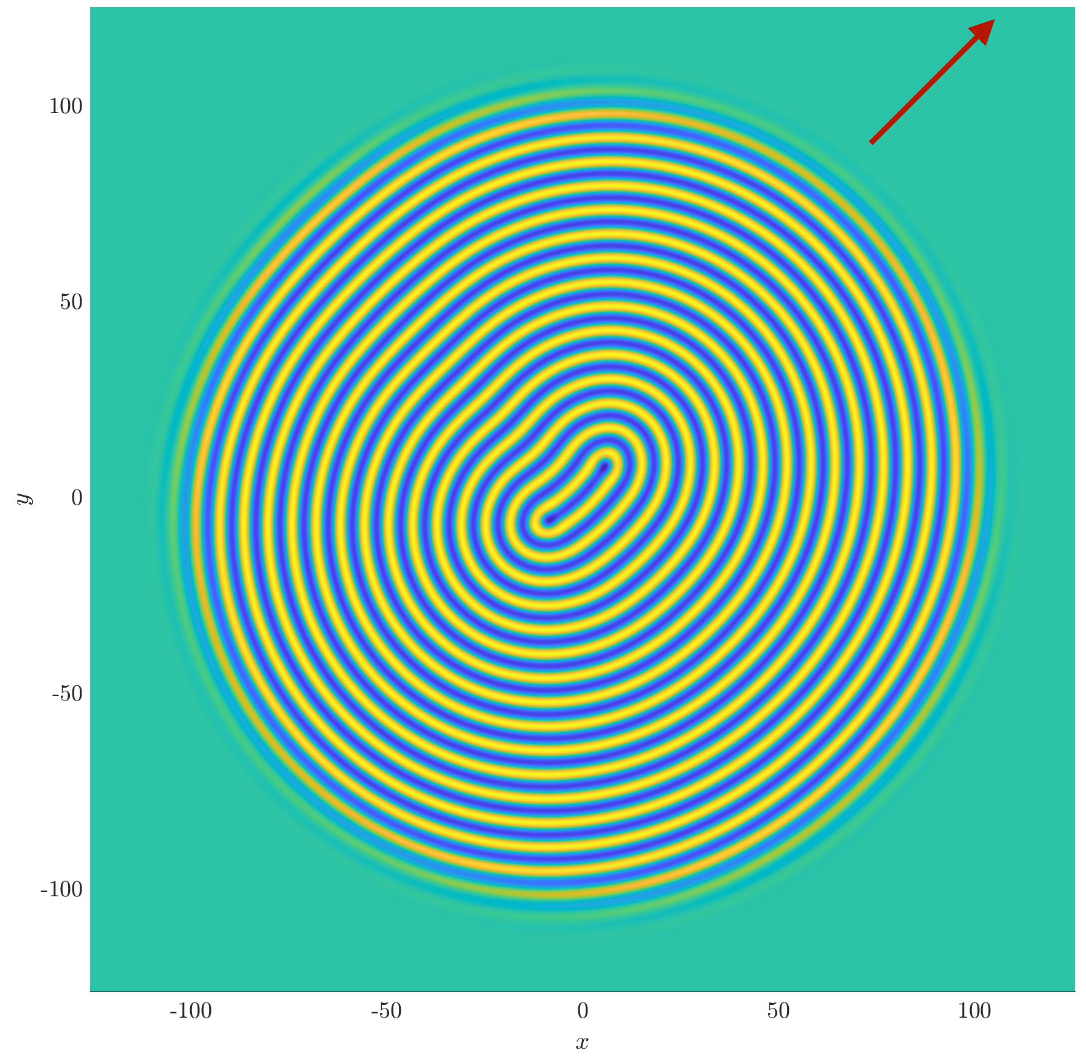


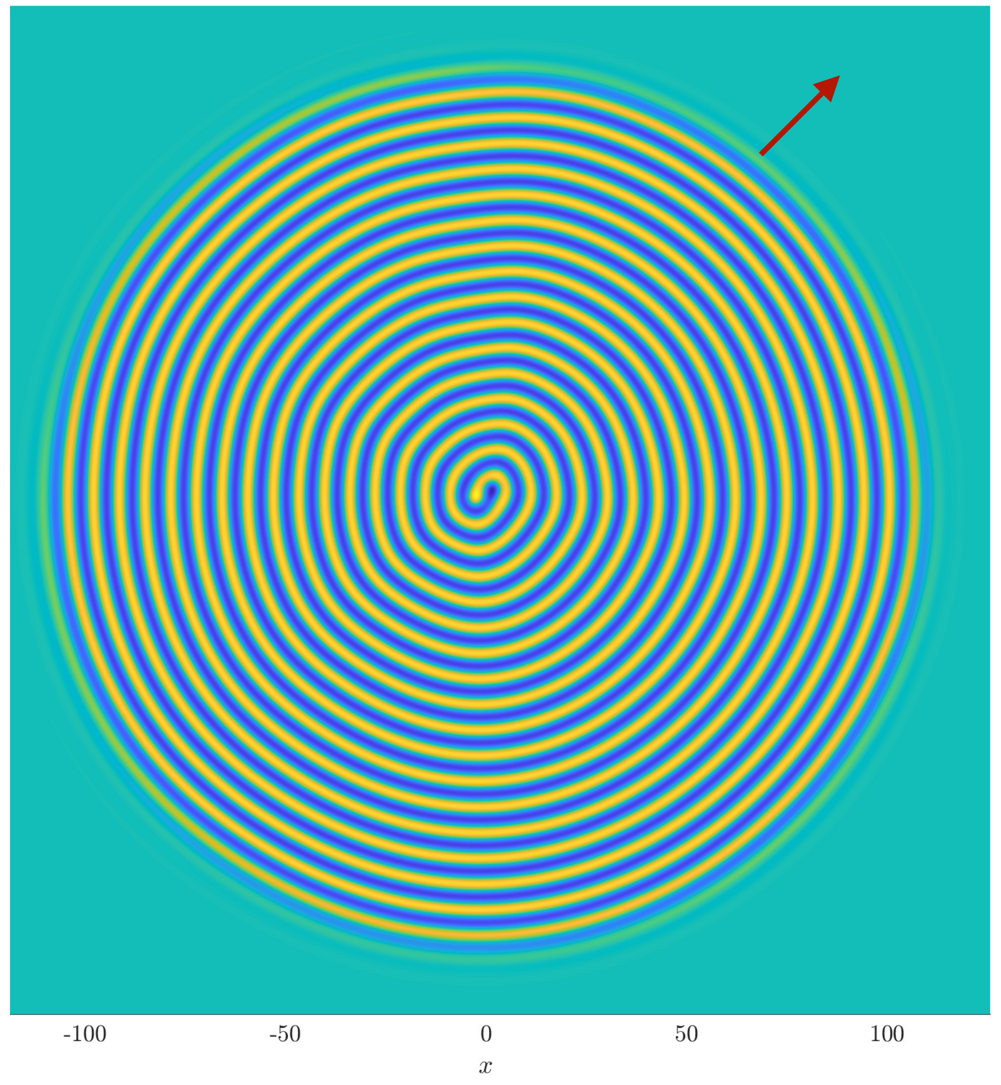
How does growth and heterogeneity control pattern formation?

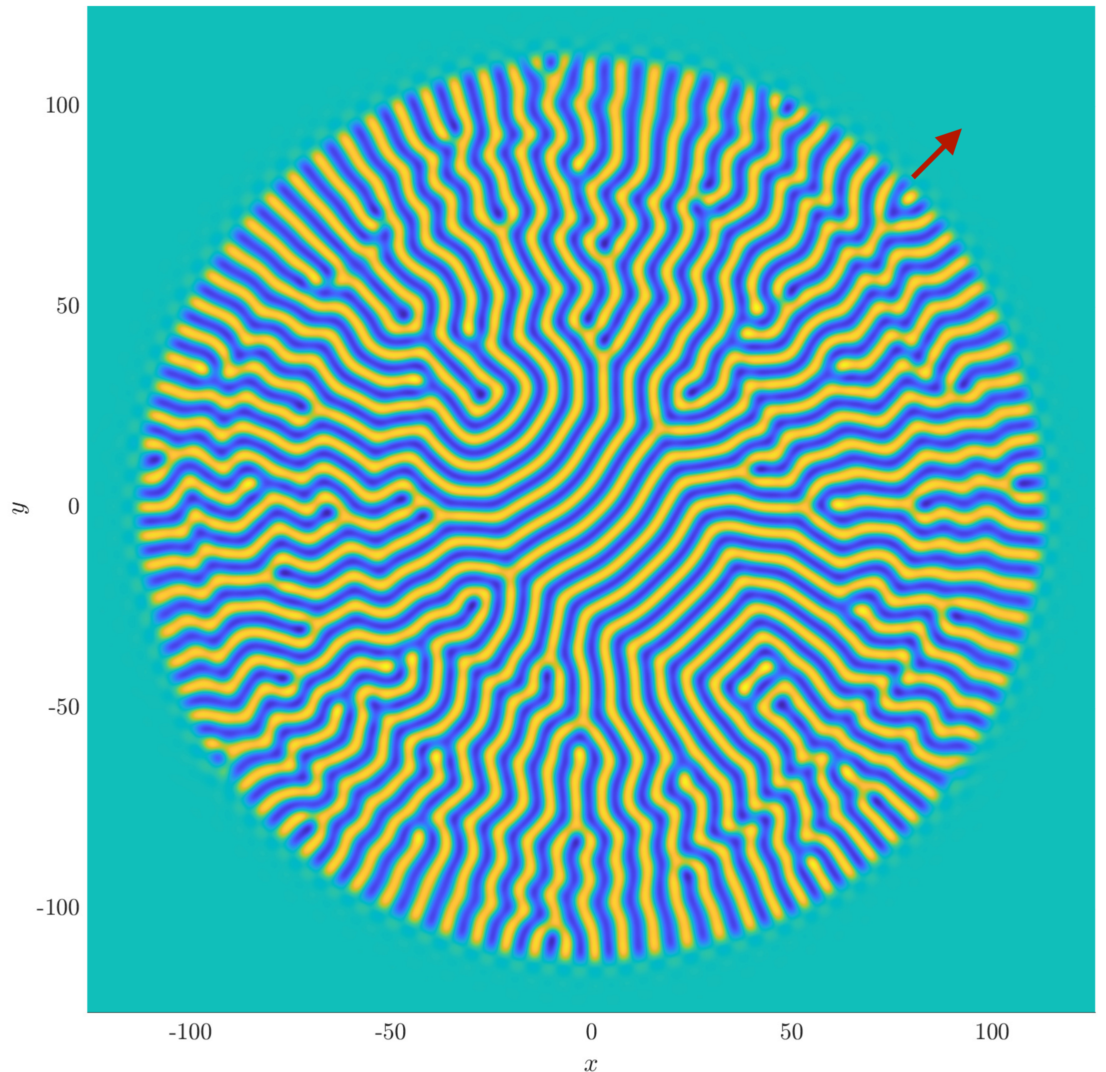


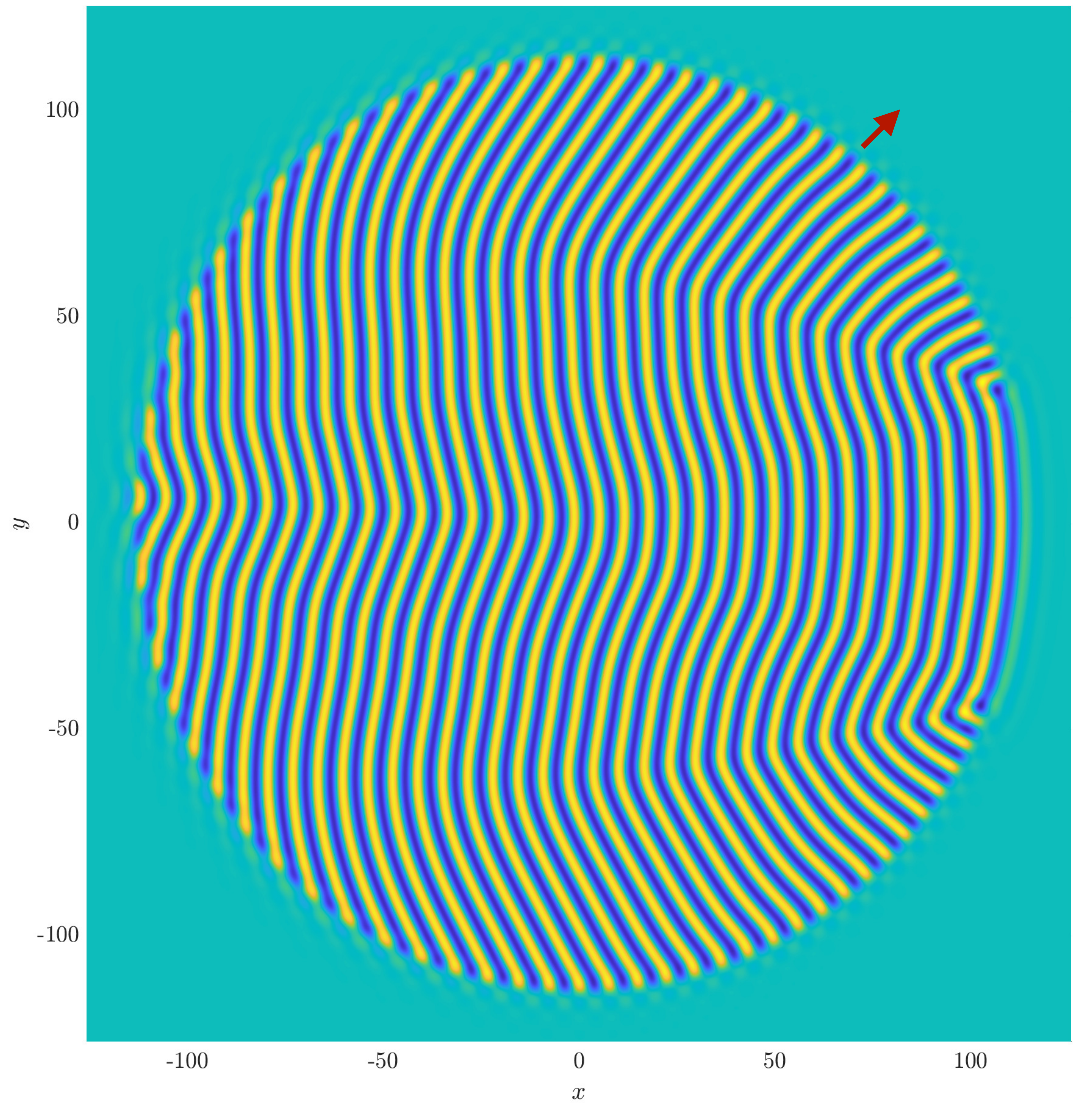








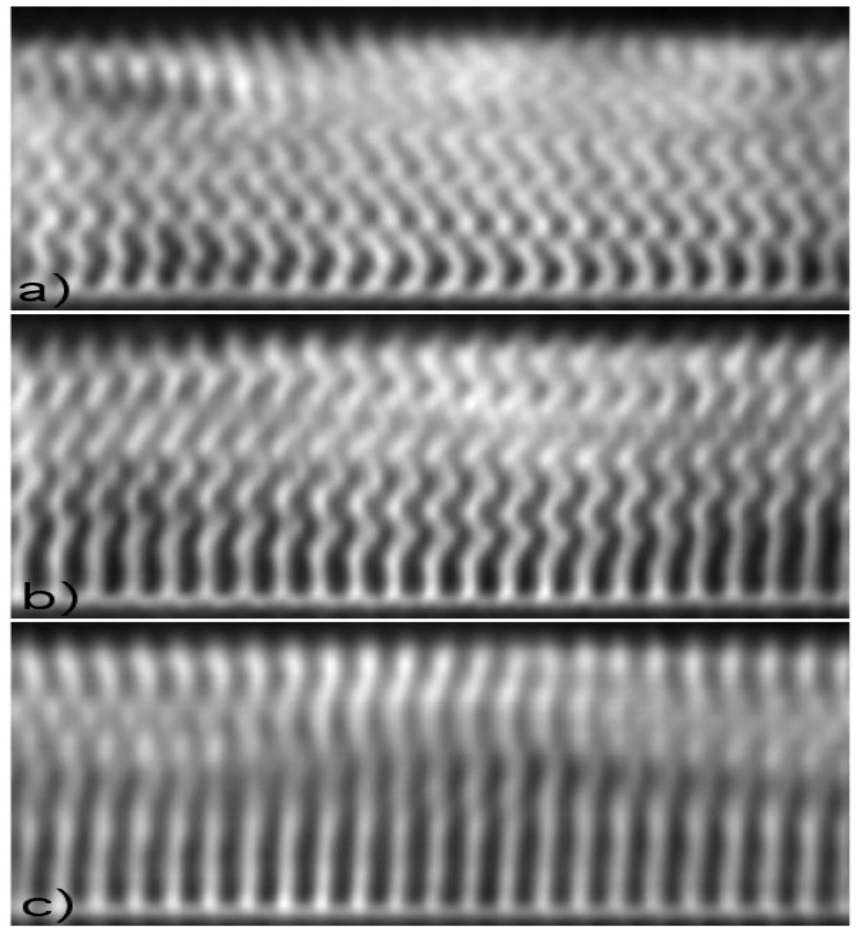




Growth and patterns

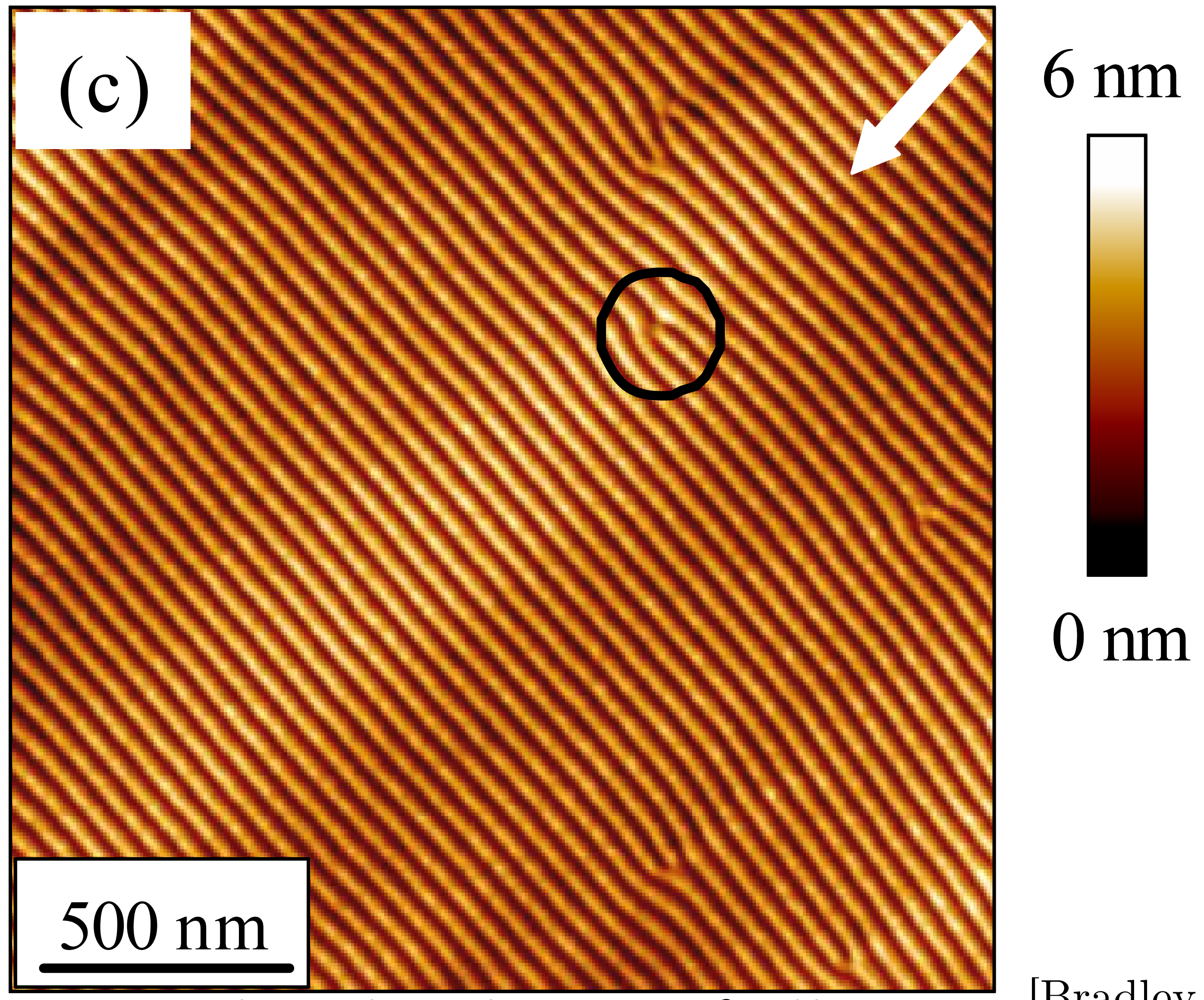
Examples in both natural and experimental systems

- Heterogeneities
- Growth mechanisms
- Quenching/solidification interfaces
- Deposition/reaction fronts



[Akamatsu, et.al, '04]

Quenching/solidifcation in Eutectic Lamellar Crystals

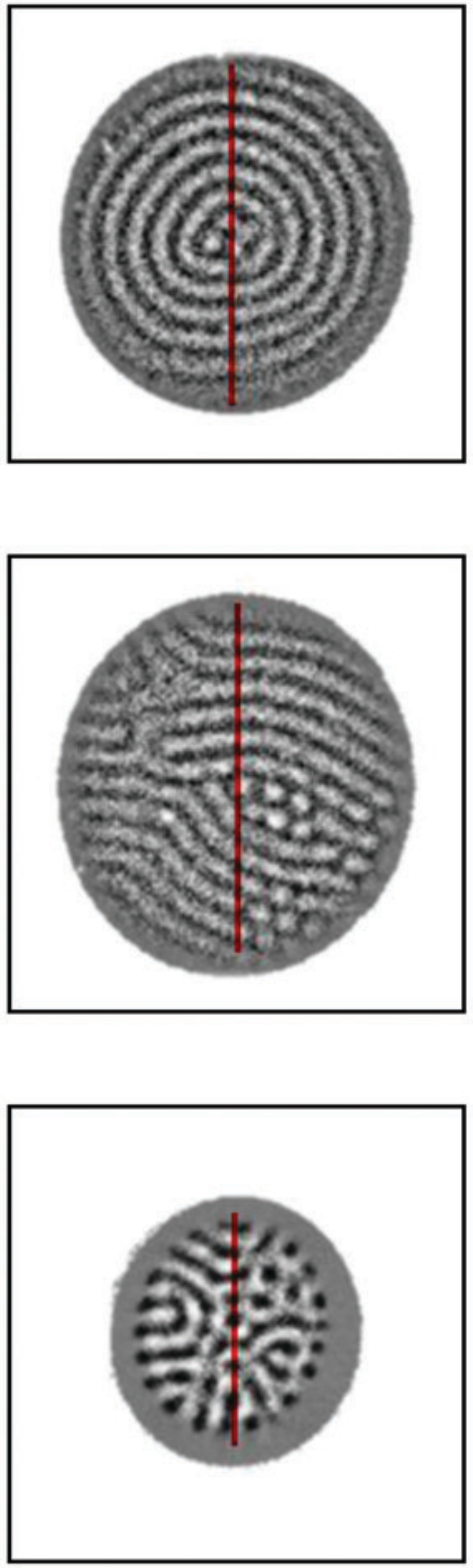


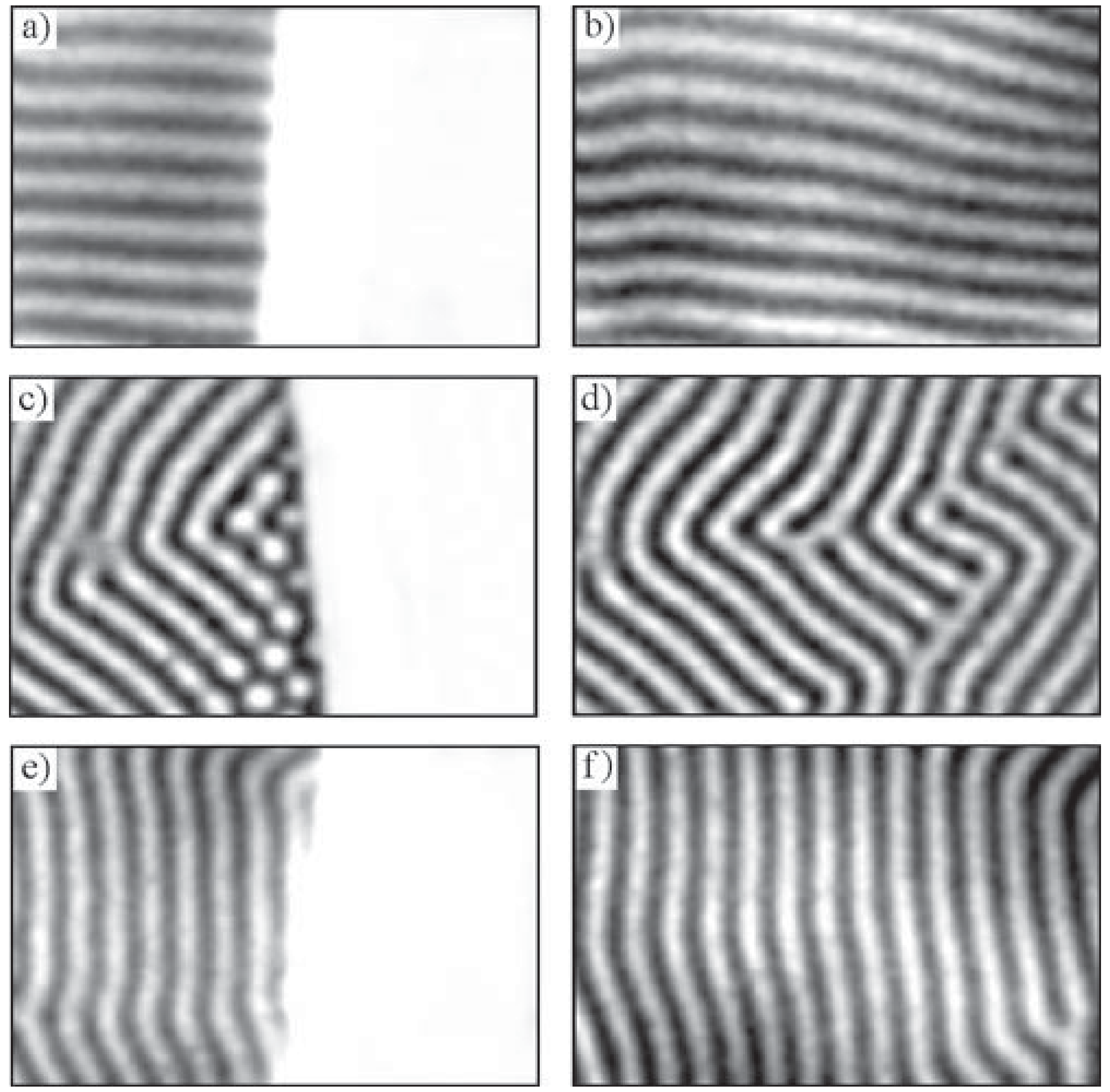
Ion bombardment of alloys

[Bradley, et.al]

Chemical precipitation

[Thomas et.al, '13]





Light-sensing CDIMA reaction-diffusion

[Miguez, et.al, '12]
[Dolnik, et. al. '19]

Outline: Fronts and slow parameter ramps

Slow parameter ramps control front formation through various dynamic bifurcations, depending on ramp speed.

1. Motivating Examples: coherent structures in presence of slow ramps
2. Fronts in Allen-Cahn model
   - i) Moving parameter ramp: absolute/convective instability, and slow passage through a fold
   - ii) Stationary parameter ramp, Painlevé-II and slow passage through a pitchfork bifurcation
   - iii) Stability (in 2 slides)
   - iv) Gluing the two regimes: Painlevé-II with drift
3. Homogeneous Ramps: fronts and patterns
4. Outlook

Collaborators

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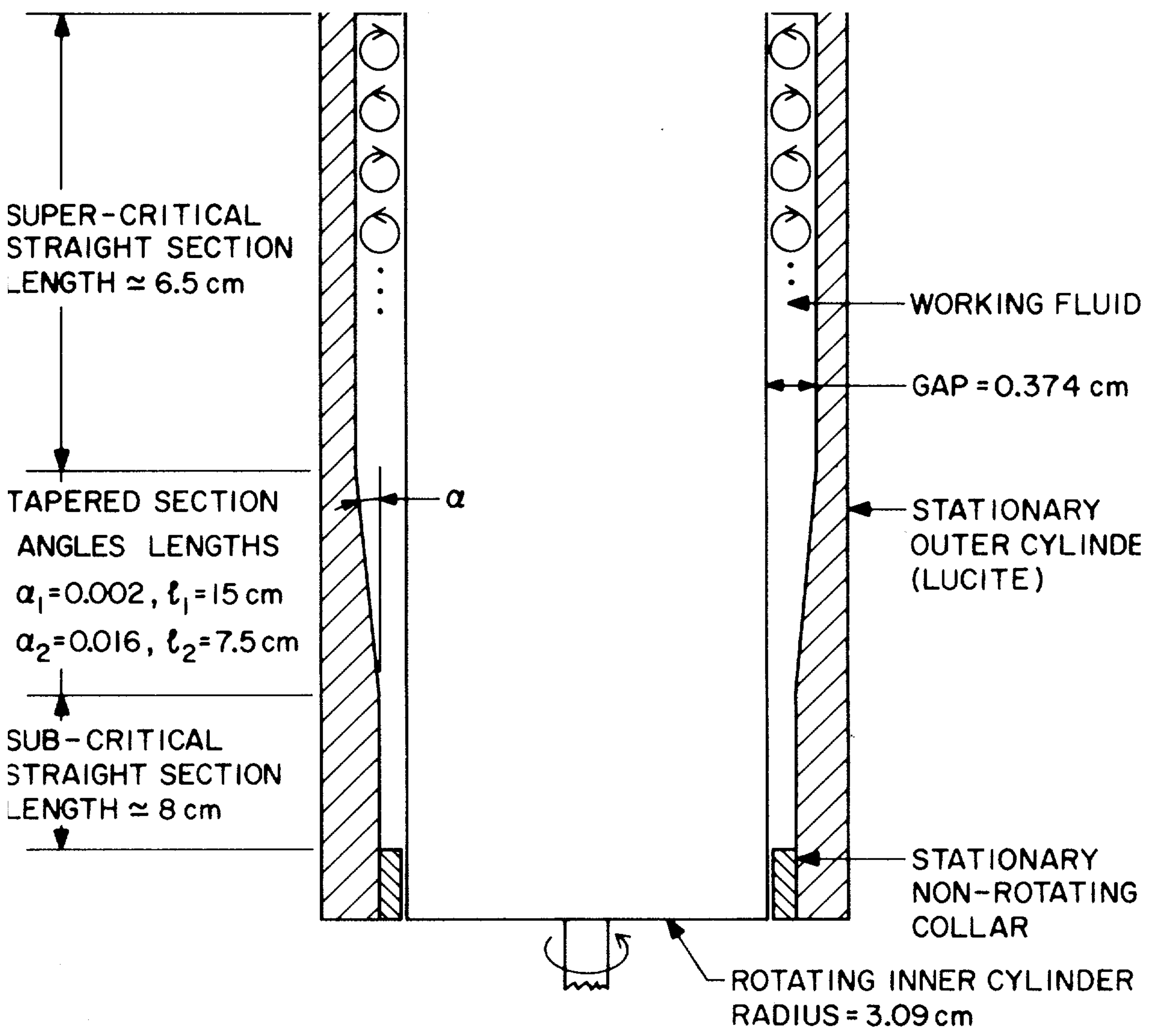
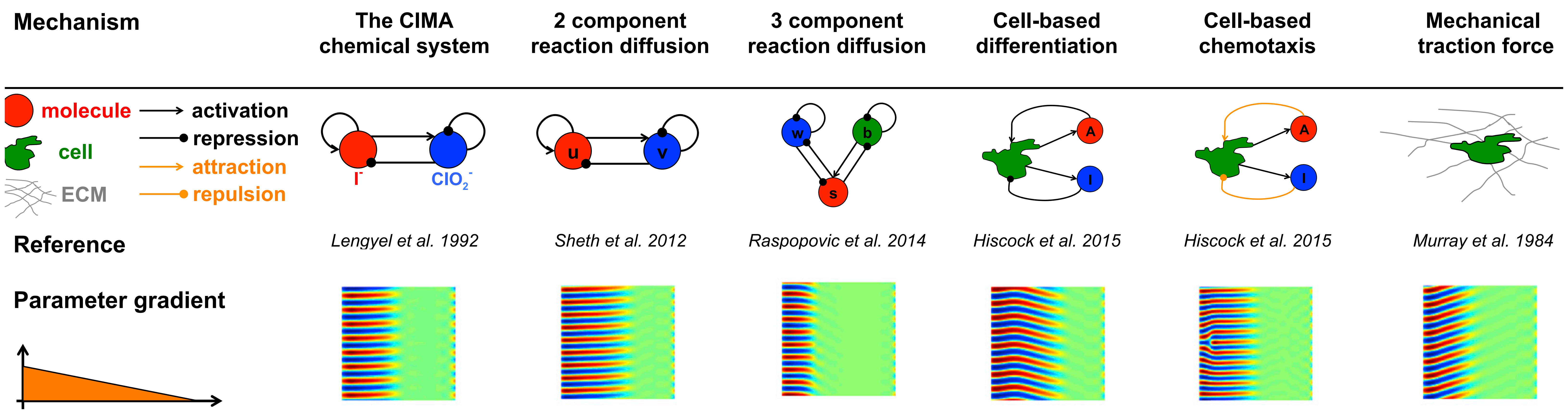
Slow ramps and patterns



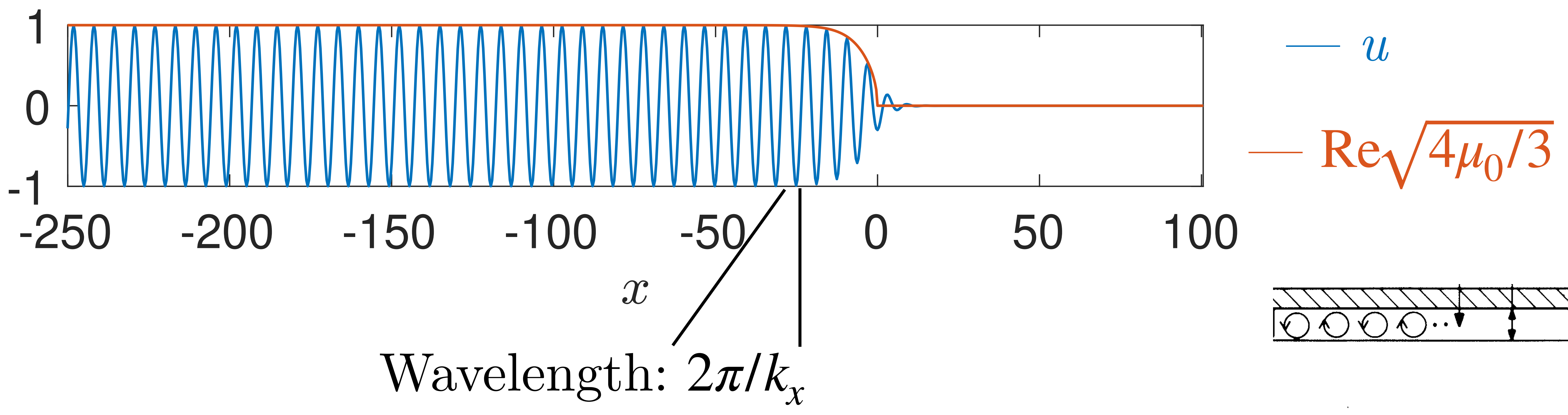
Taylor-Couette Flow
[Cannell 1983]



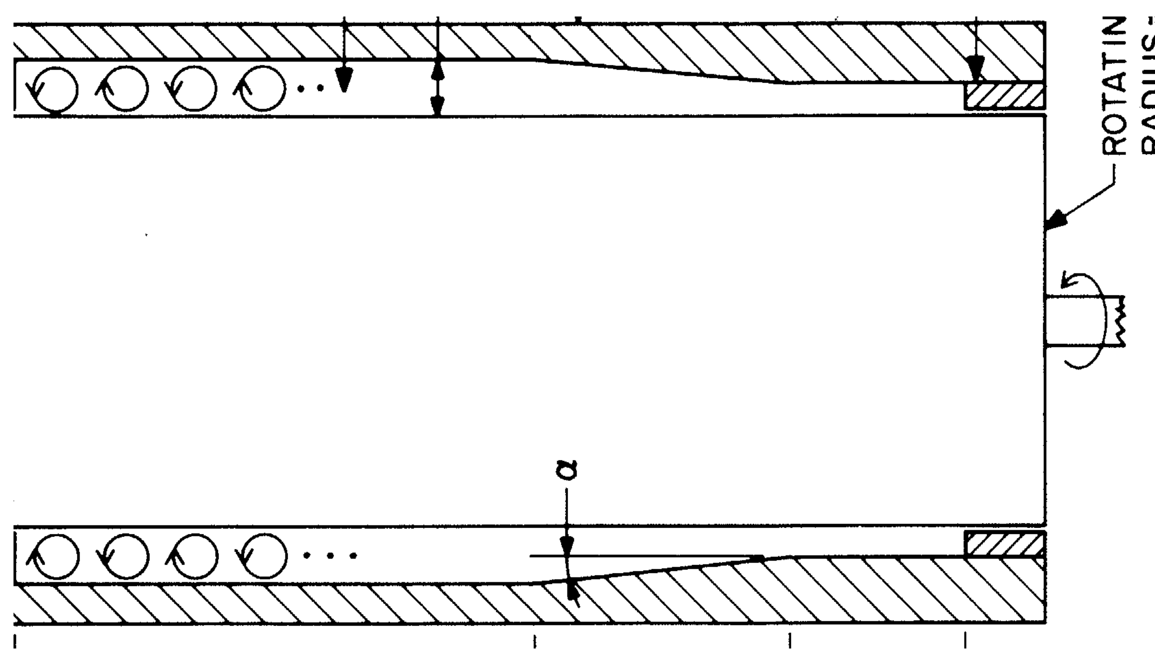
Biologically inspired models
[Hiscock & Megason 2015]

Also [Rehberg, Riecke, 1987], [Chomaz, Couraion], [Pomeau,Zaleski]

Swift-Hohenberg

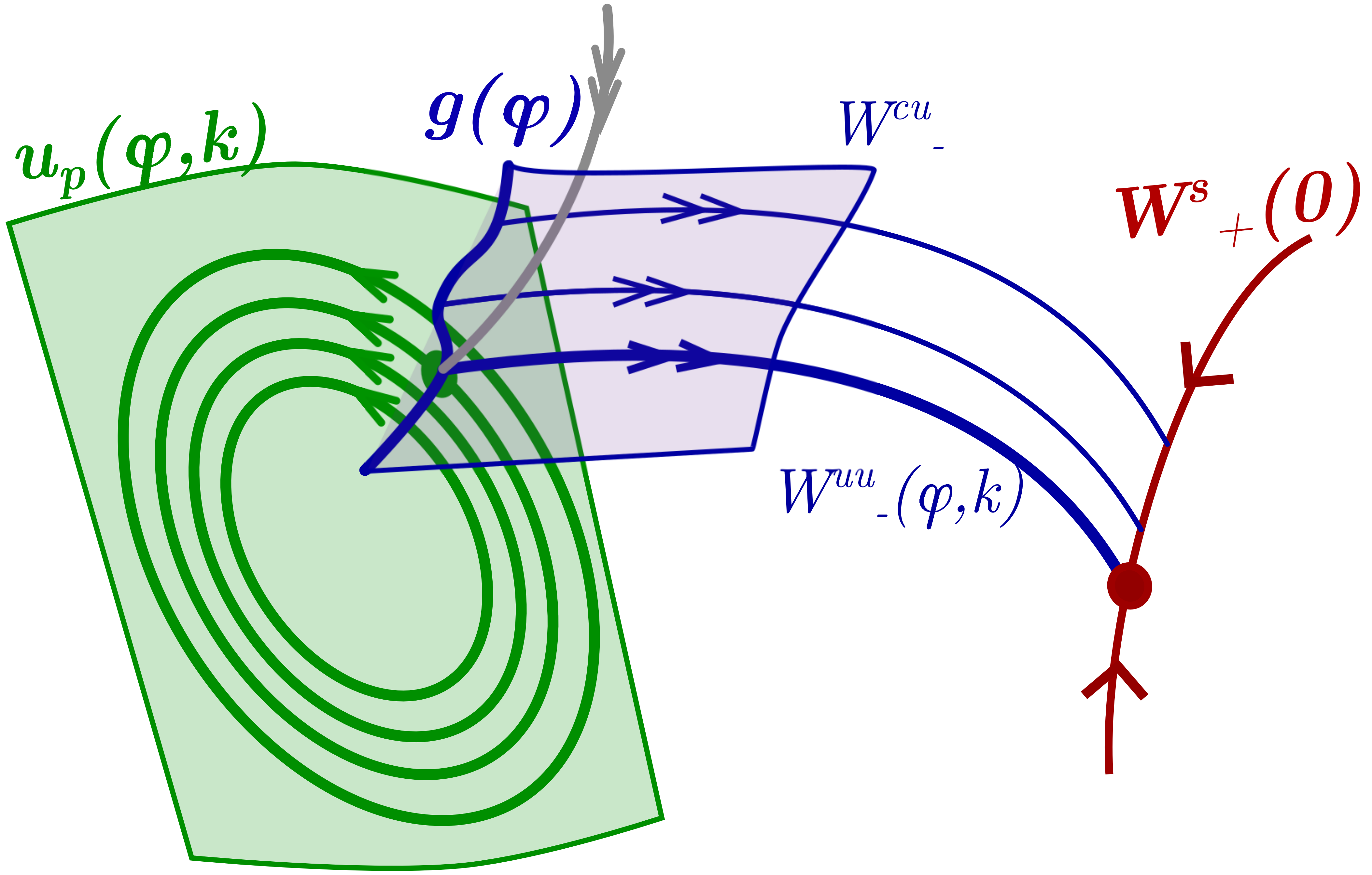


$$u_t = -(1+\partial_x^2)^2 u + \mu(x)u - u^3$$
$$\mu(x) = -\mu_0 \tanh(\epsilon x)$$



Ramp slope: ϵ

Front solutions: $u(x) \to \sqrt{4\mu_0/3}\cos(k_x x + \phi) + O(\mu_0)$ as $x \to -\infty$
$u(x) \to 0$ as $x \to +\infty$



Swift-Hohenberg

$$u_t = -(1+\partial_x^2)^2 u + \mu(x) u - u^3$$
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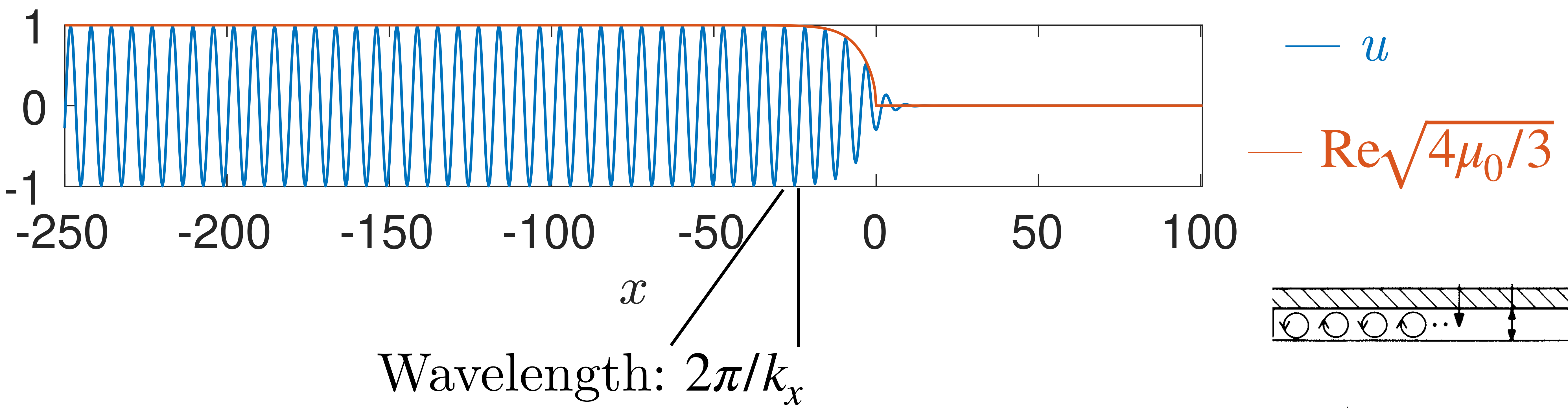
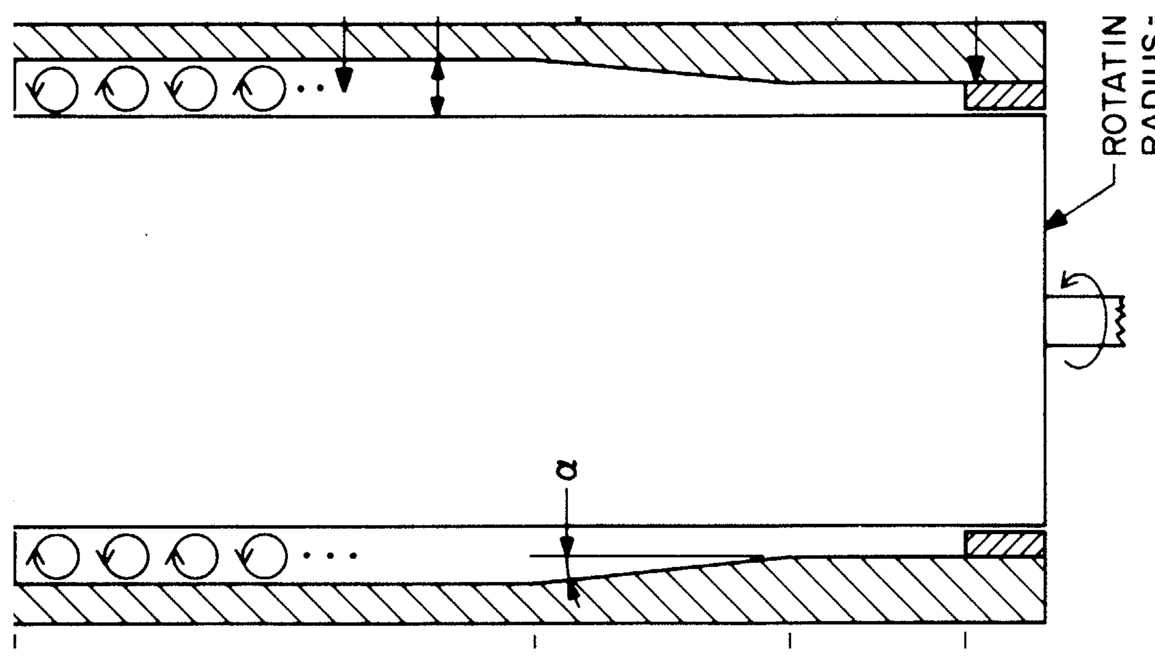
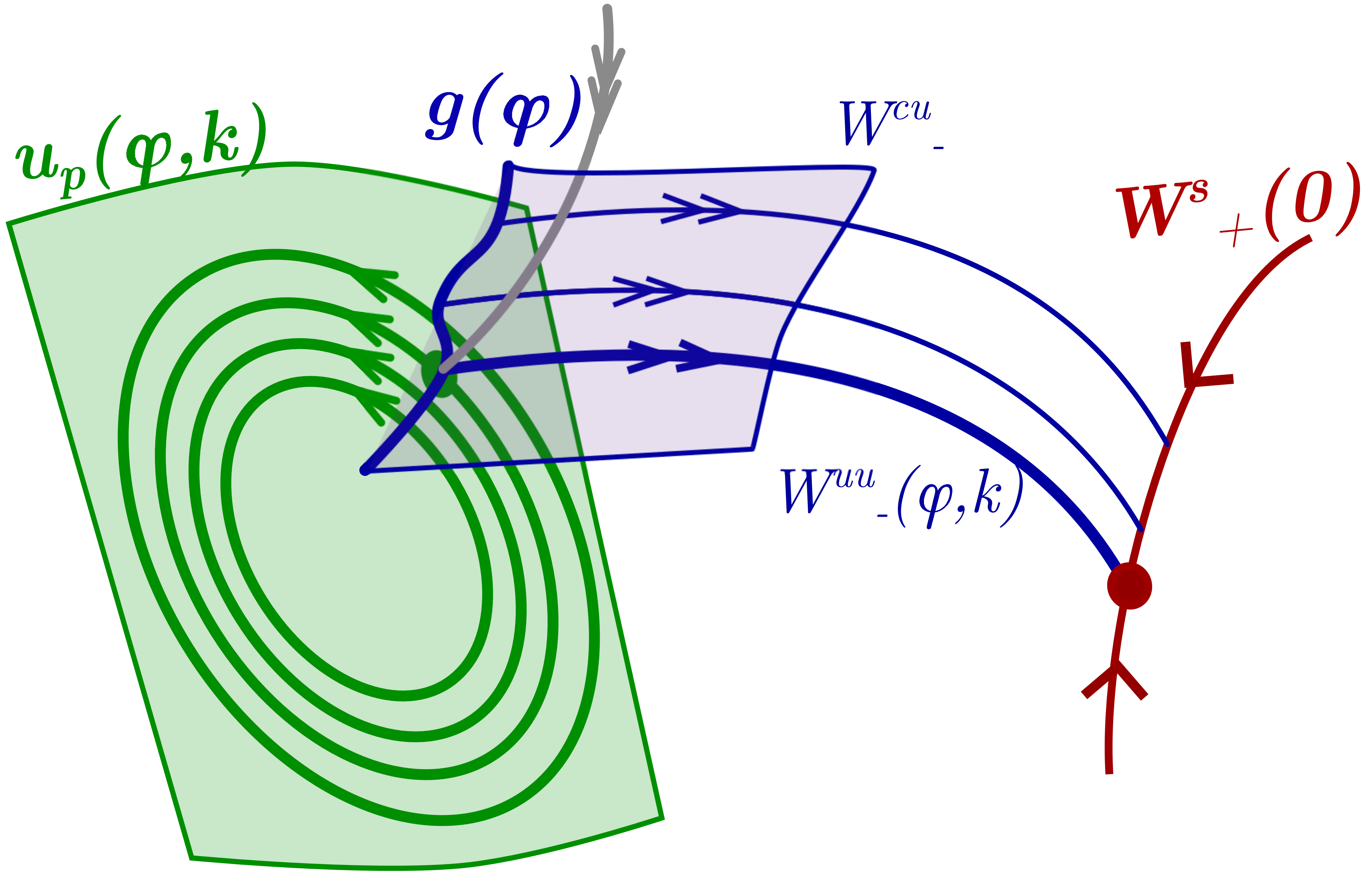


Ramp slope: ϵ

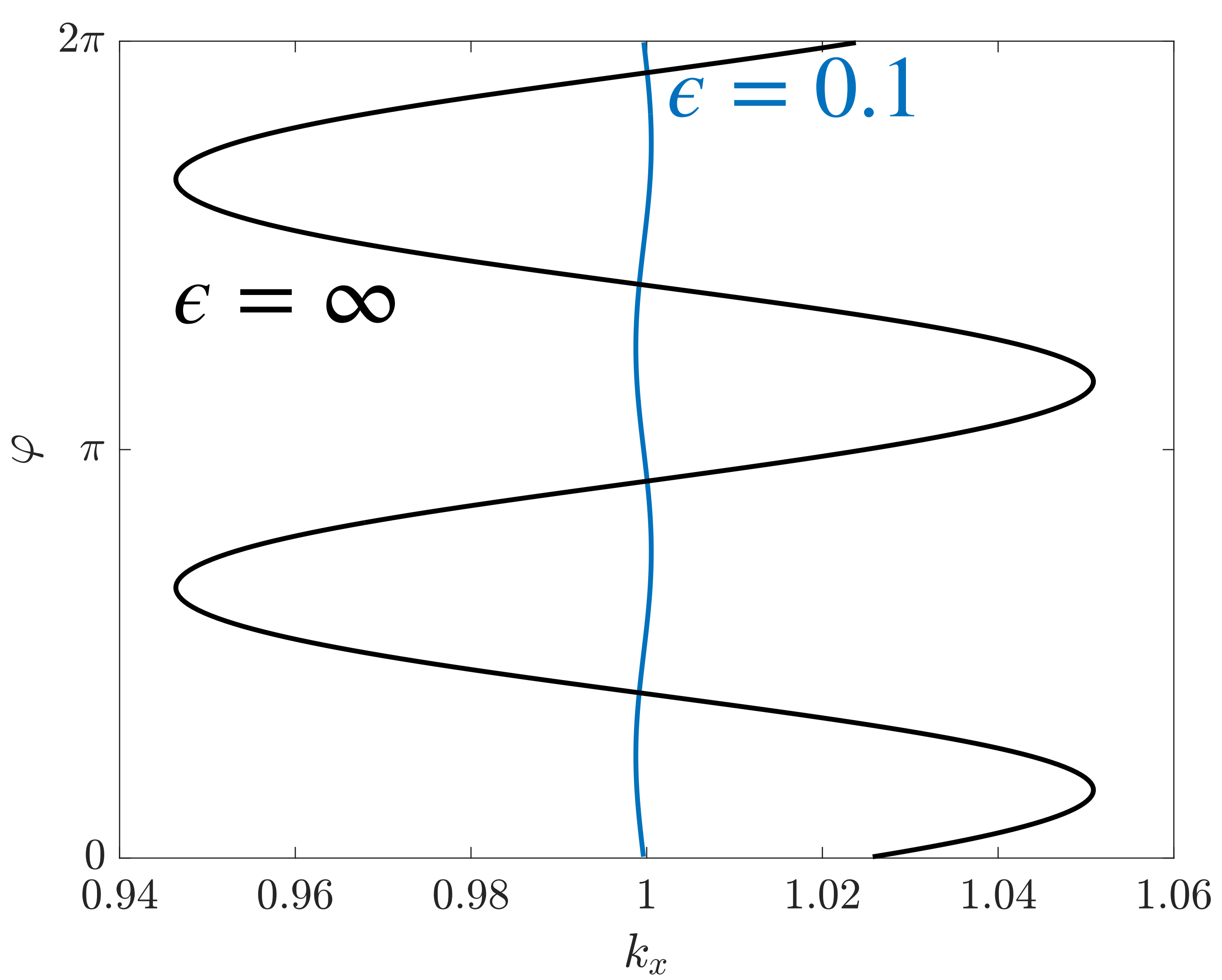


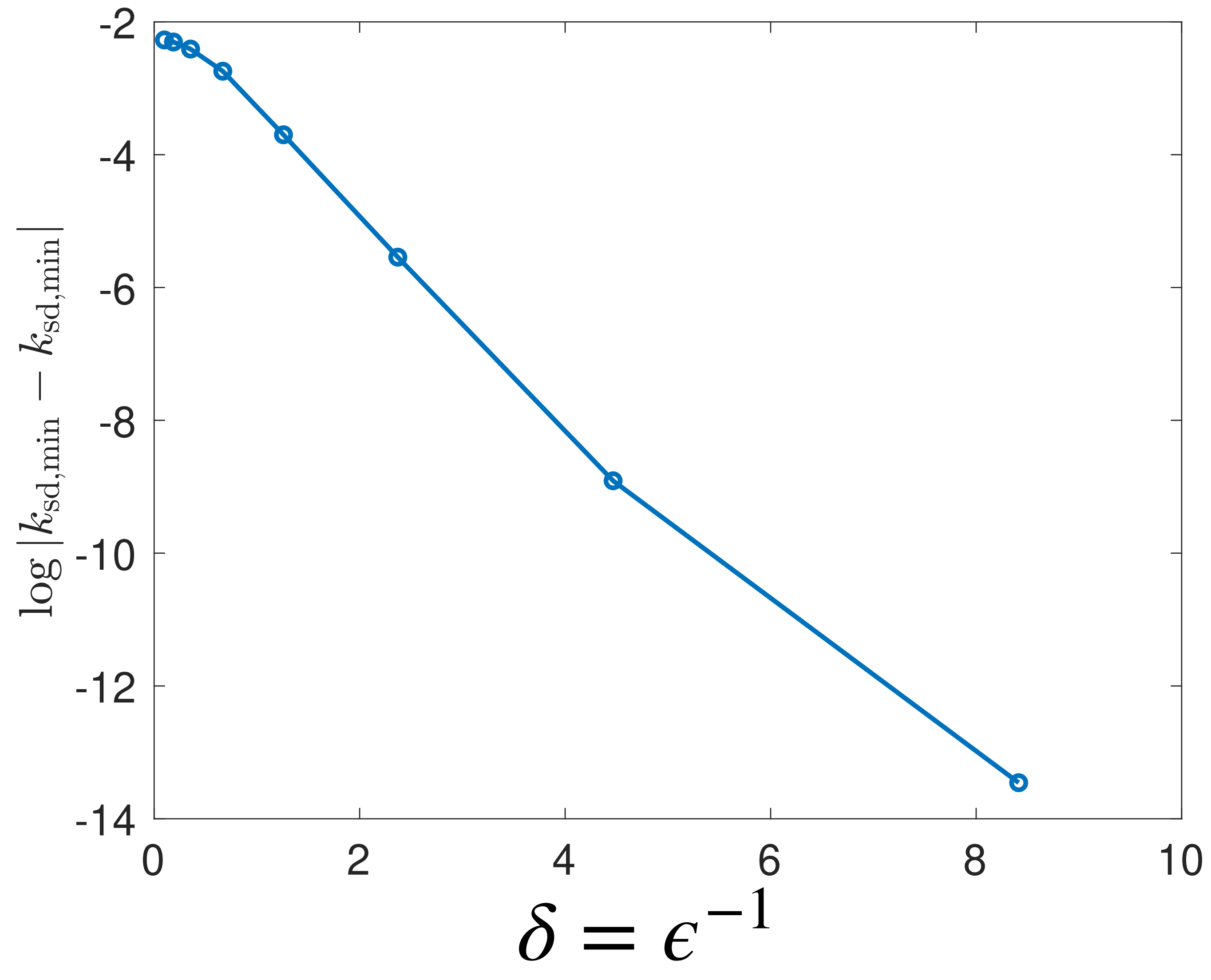
Front solutions: $u(x) \to \sqrt{4\mu_0/3}\cos(k_x x + \phi) + O(\mu_0) \quad$ as $x \to -\infty$

$u(x) \to 0$ as $x \to +\infty$



<u>$c = 0$:</u> Strain-Displacement relations, $\epsilon = \infty$ -> [Weinburd, Morrissey, Scheel]

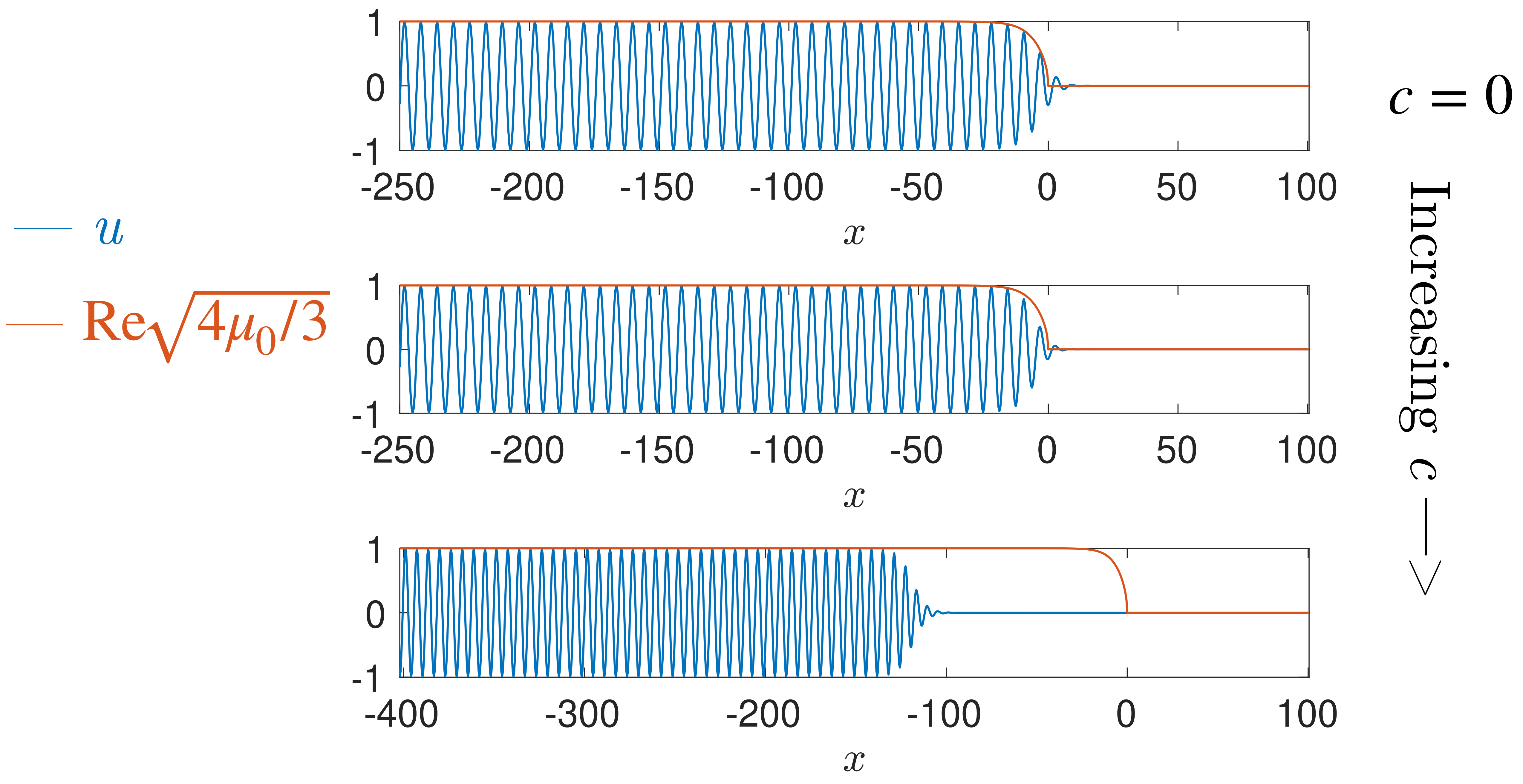




Swift-Hohenberg - moving ramp

$$u_t = -(1+\partial_x^2)^2 u + \mu(x-ct)u - u^3$$

$$\mu(\xi) = -\mu_0 \tanh(\epsilon\xi), \qquad \xi := x - ct$$



Front solutions with:

$u(\xi, \omega t) \to \sqrt{4\mu_0/3}\cos(k_x\xi + \omega t) + O(\mu_0)$, as $x \to -\infty$, frequency $\omega = ck_x$

$u(\xi, \omega t) \to 0$ as $x \to +\infty$

$u(\xi, \tau + 2\pi) = u(\xi, \tau)$

$\epsilon = +\infty$ —> [RG, Scheel, '18, '23]

Swift-Hohenberg - moving ramp

$$u_t = -(1+\partial_x^2)^2 u + \mu(x-ct)u - u^3$$

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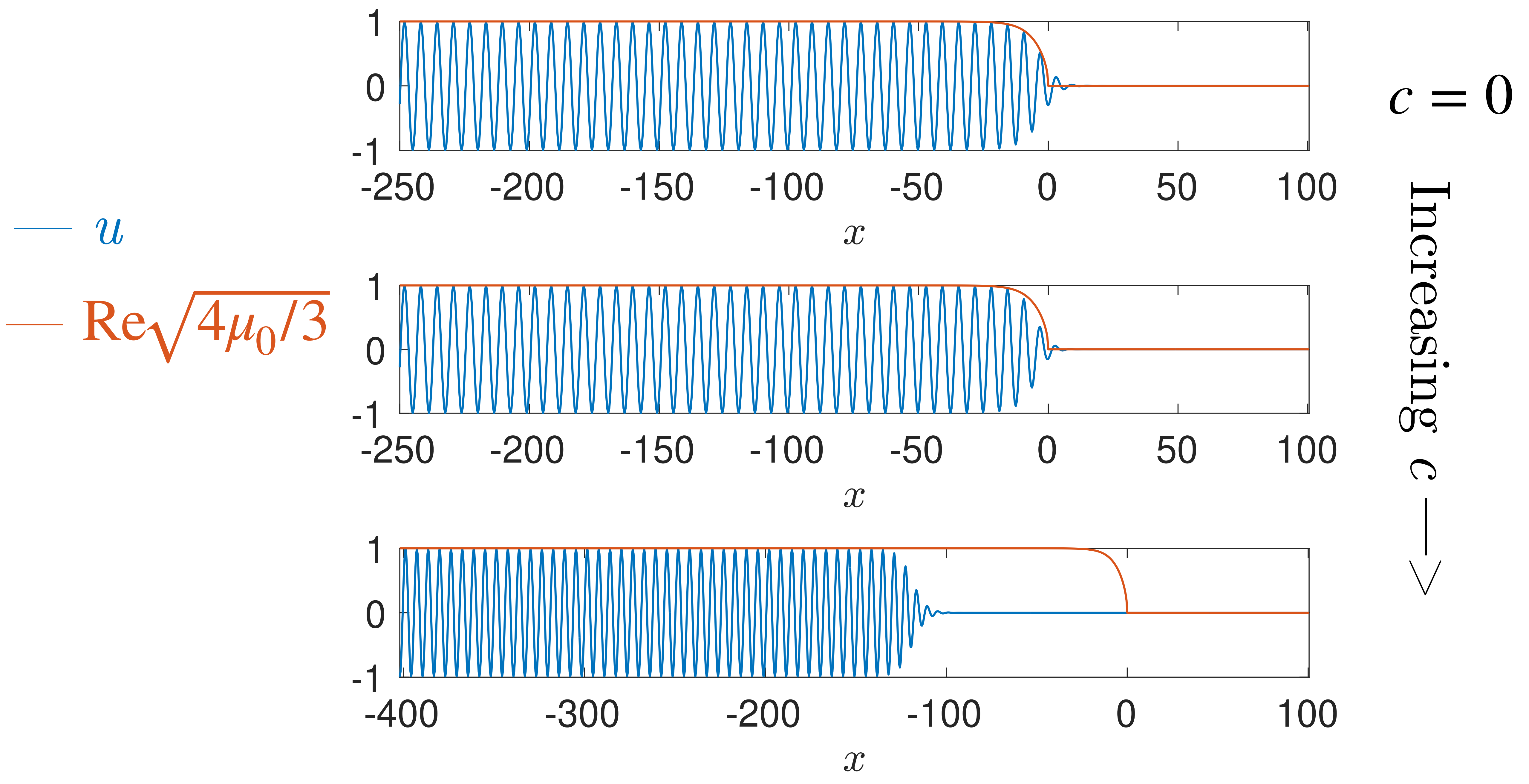

Front solutions with:

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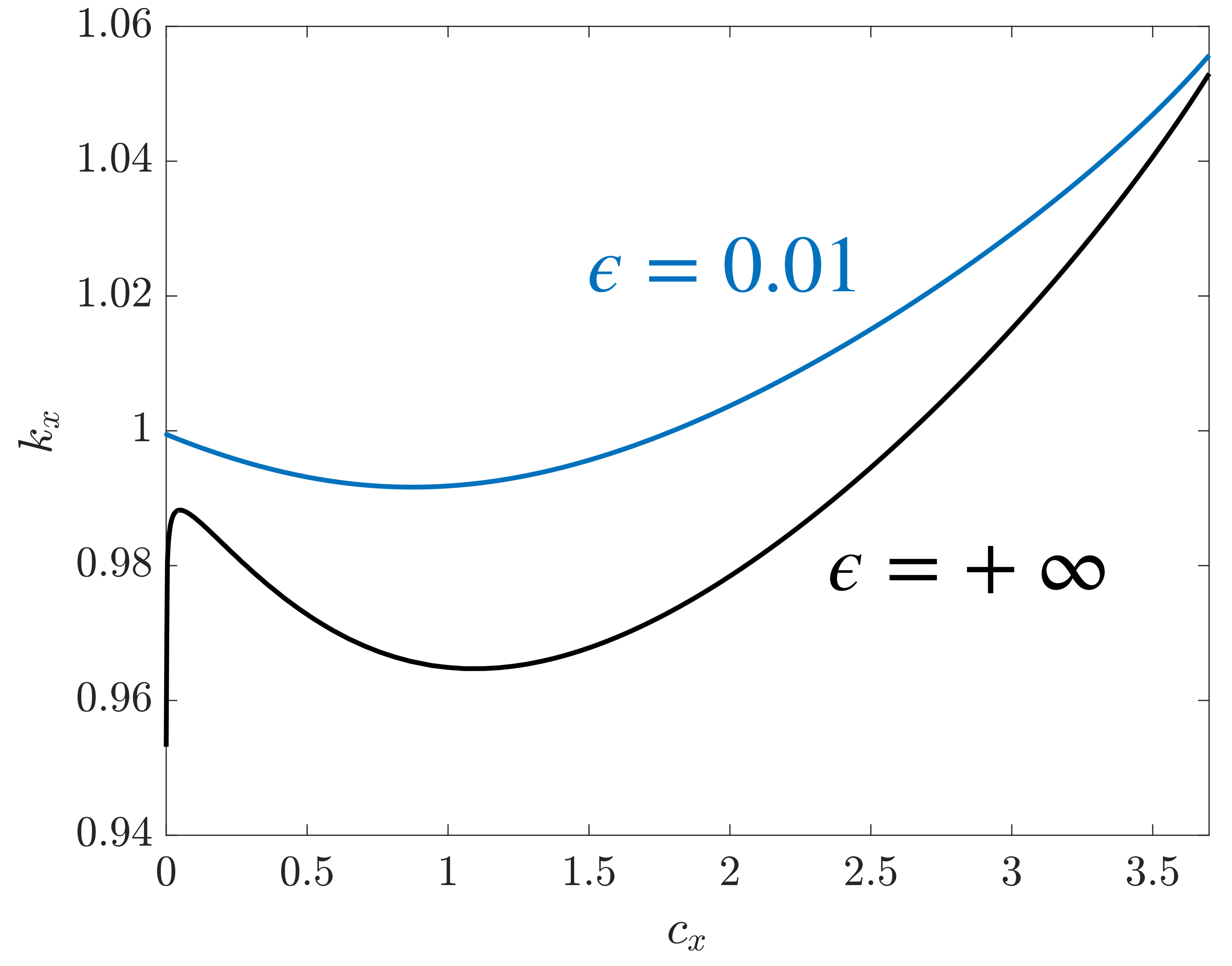
$u(\xi,\omega t) \to 0$ as $x \to +\infty$

$u(\xi,\tau+2\pi) = u(\xi,\tau)$

$c > 0$: *wavenumber selection*



$\epsilon = +\infty$ —> [RG, Scheel, '18, '23]

First step: Allen-Cahn model

$$u_t = u_{xx} + \mu(x-ct)u - u^3, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+$$
$$\mu(\xi) = -\tanh(\epsilon\xi), \quad 0 < \epsilon \ll 1$$

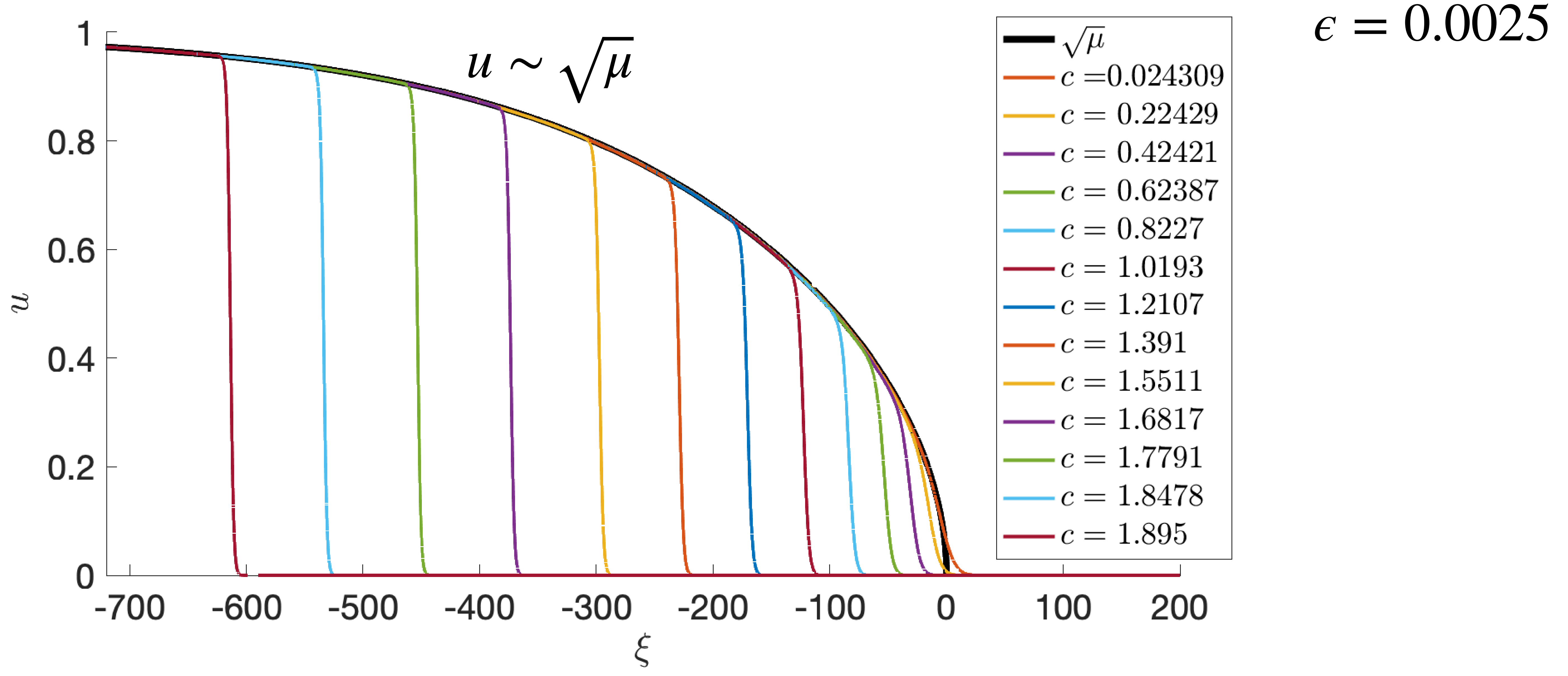
Ramp speed: $c \geq 0$ Ramp slope: ϵ

Travelling wave solutions: $u(x,t) = u(x-ct)$, $\xi := x - ct$

$$0 = u_{\xi\xi} + cu_\xi + \mu u - u^3,$$
$$\mu_\xi = -\epsilon(1-\mu^2)$$

$\lim_{\xi\to-\infty} u(\xi) = 1,$ $\lim_{\xi\to+\infty} u(\xi) = 0$

$\lim_{\xi\to-\infty} \mu(\xi) = 1,$ $\lim_{\xi\to+\infty} \mu(\xi) = -1$



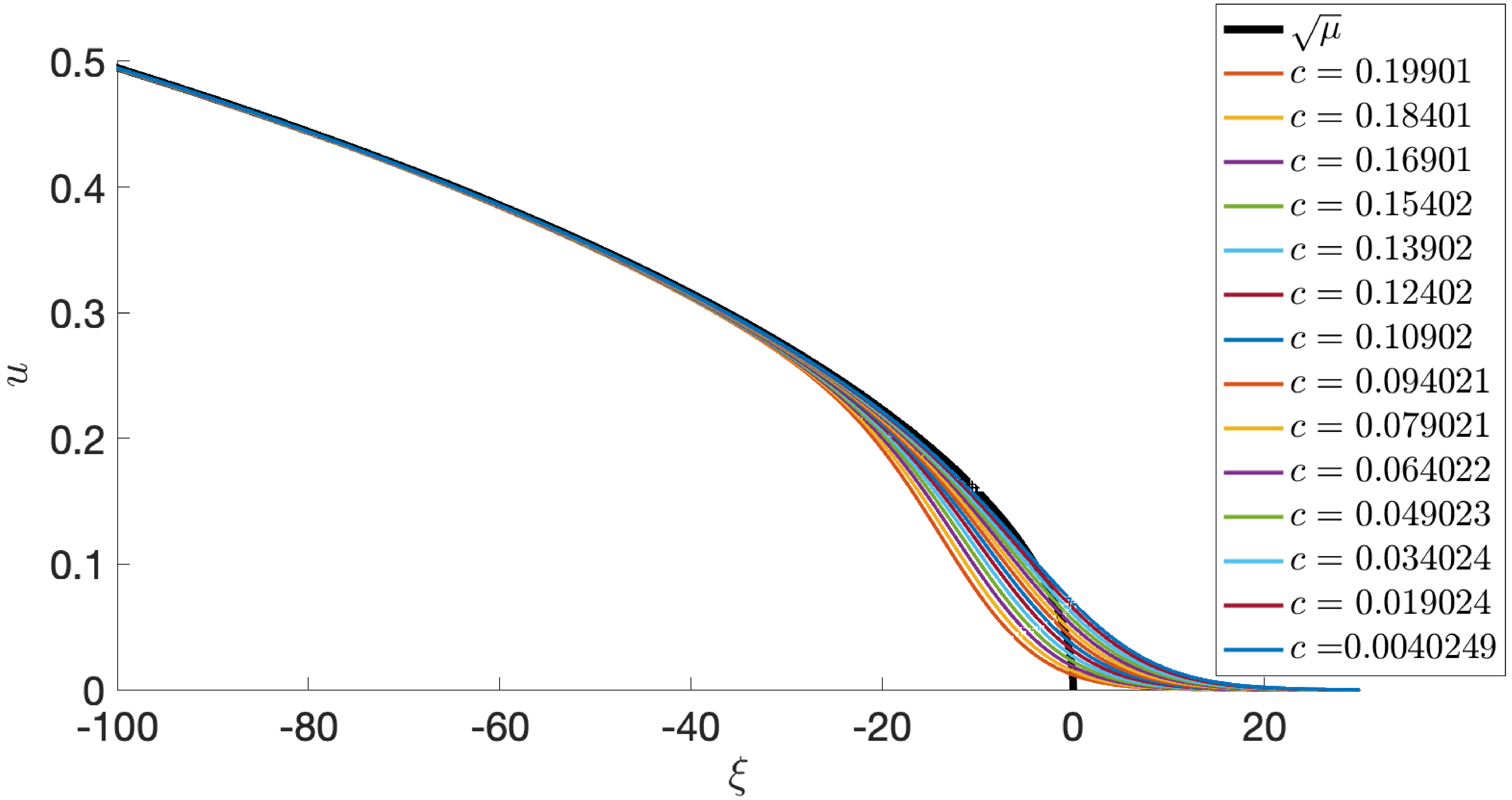
$\mu > 0, \xi < 0$

$u = 0$ temporally unstable,

$\mu < 0$, $\xi > 0$,

$u = 0$ temporally stable

Two generic regimes



$c = \mathcal{O}_\epsilon(1), \qquad 0 < c < 2$

- Fronts do not (!) take off at instantaneous stability transition $\mu = 0$
- Leading order: front interface governed by μ transition between convective and absolute instability
- Slow ramp induces a further *delay*, controlled by *slow passage through a fold* (of eigenspaces)

$c \sim 0$

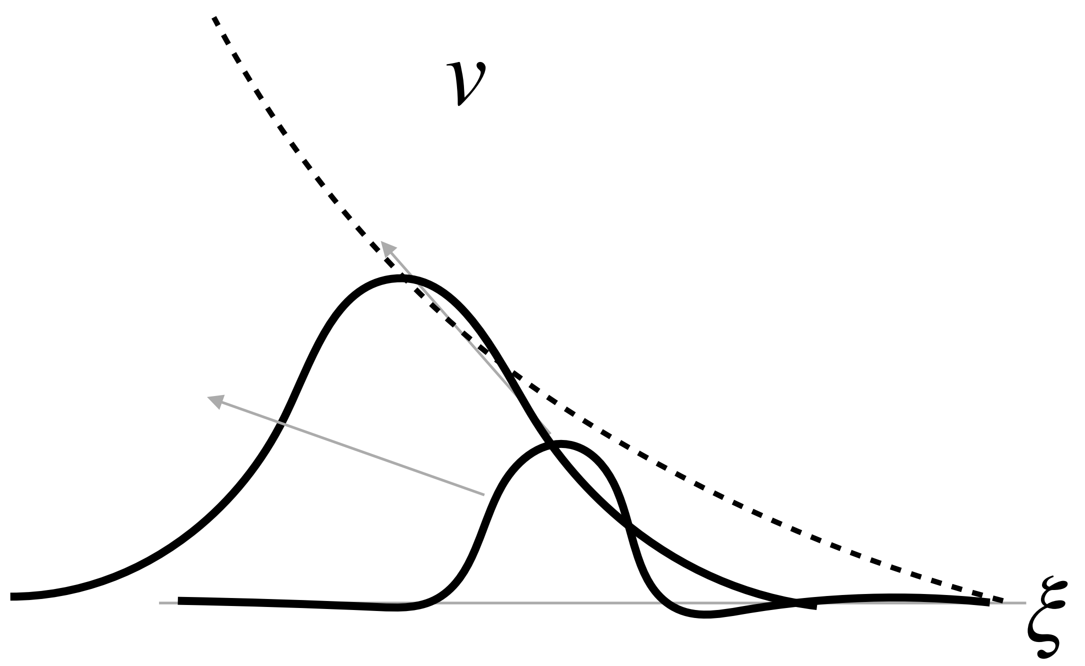
- Diffusive tail of front into $\mu < 0$ region
- *Slow passage through a pitchfork bifurcation* controls front interface
- Hastings-McLeod *connecting* solution of Painlevé-II equation gives “inner” solution

(Technically $c \gg \epsilon^{1/3}$, $\qquad c \ll \epsilon^{1/3}$)

1st Regime: $c = \mathcal{O}_\epsilon(1), \qquad 0 < c < 2$

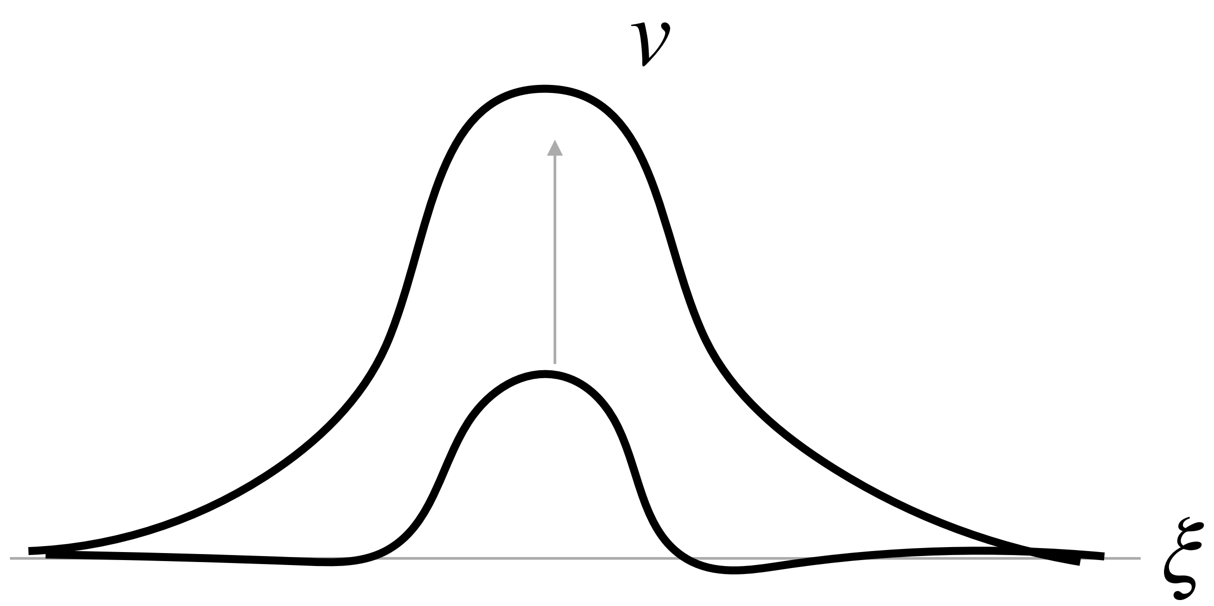
Absolute and convective instability

$v_t = v_{\xi\xi} + cv_\xi + \mu v =: L(\mu, c)v$



—> decreasing c —>

—> increasing μ —>



Convective instability $\mu < \mu_c$

Absolute instability $\mu > \mu_c$

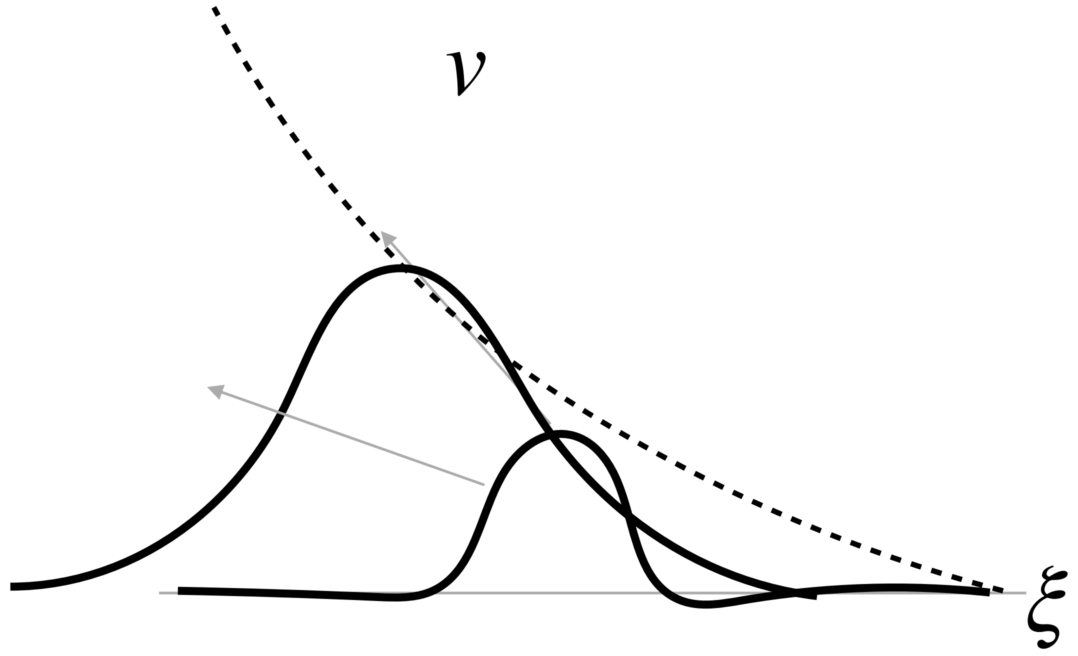
Dispersion relation $\lambda = \nu^2 + c\nu + \mu \implies$ transition at $\mu_c = c^2/4$.

set $\mu(\xi_c) = \mu_c$

1st Regime: $c = \mathcal{O}_\epsilon(1), \qquad 0 < c < 2$

Absolute and convective instability

$$v_t = v_{\xi\xi} + c v_\xi + \mu v =: L(\mu, c) v$$



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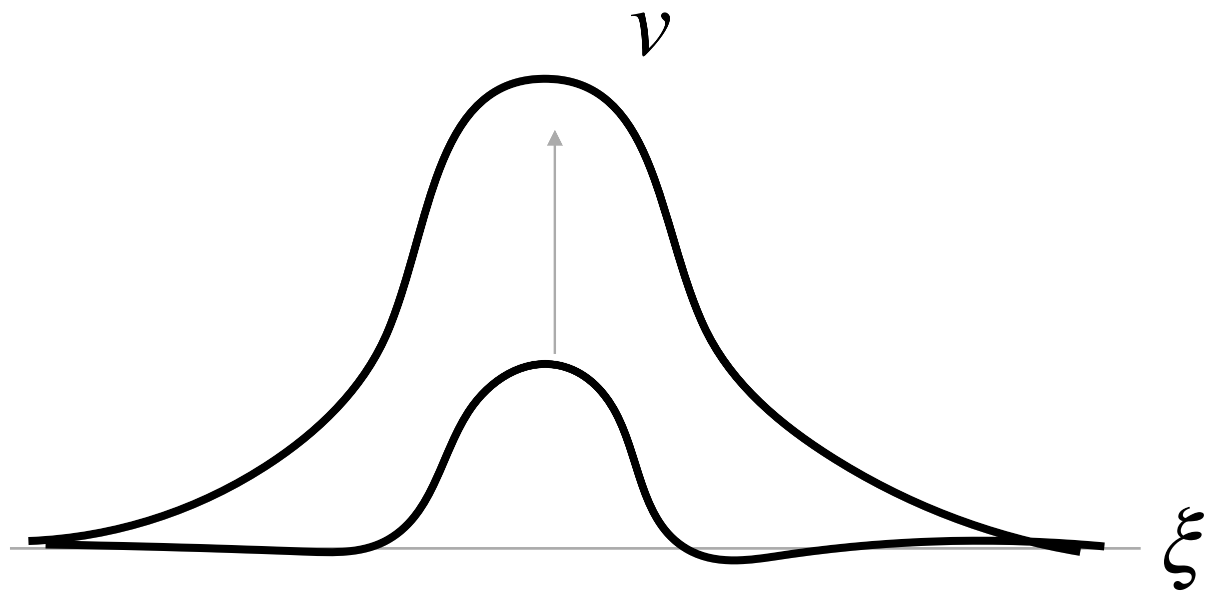

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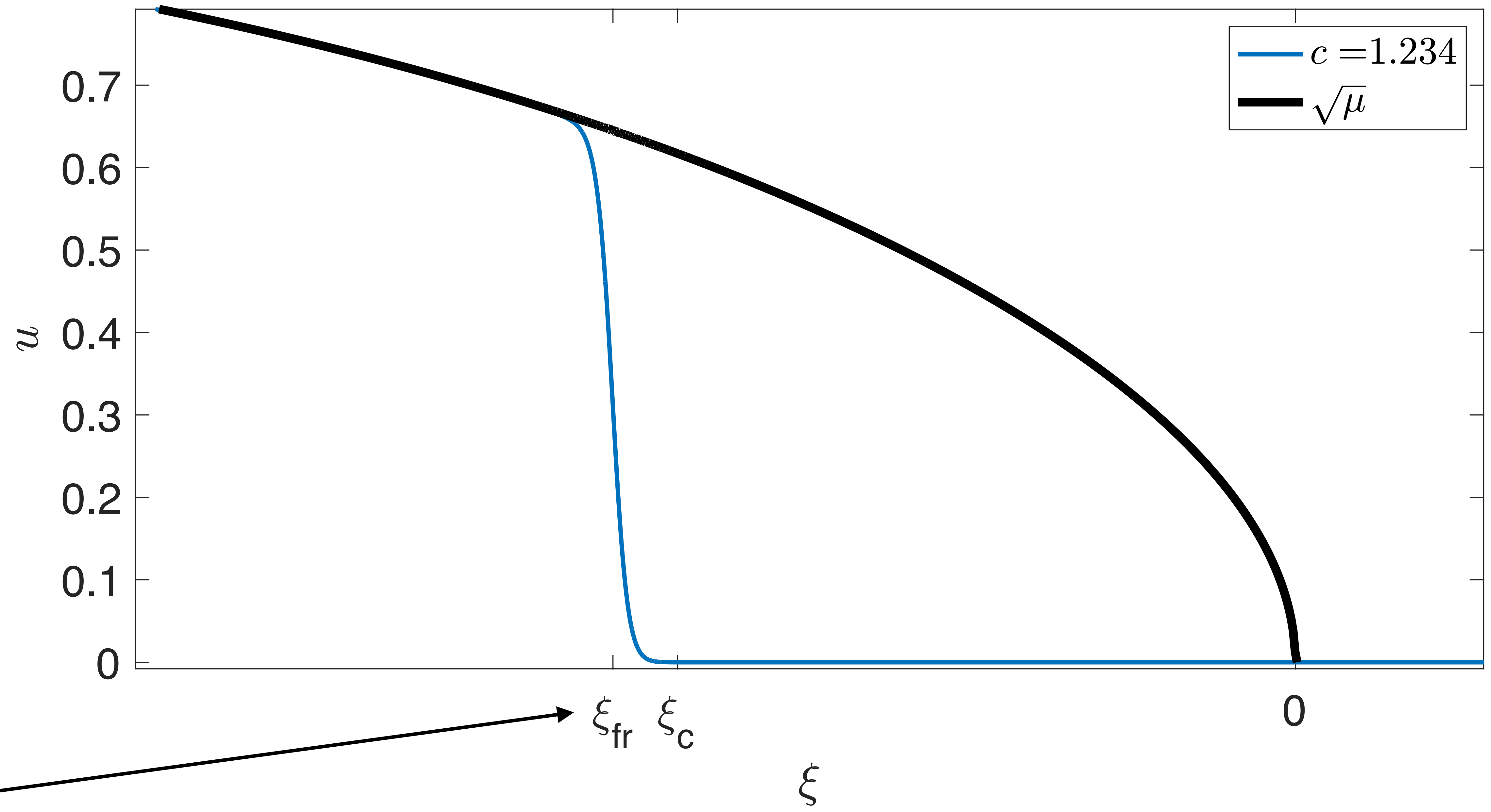
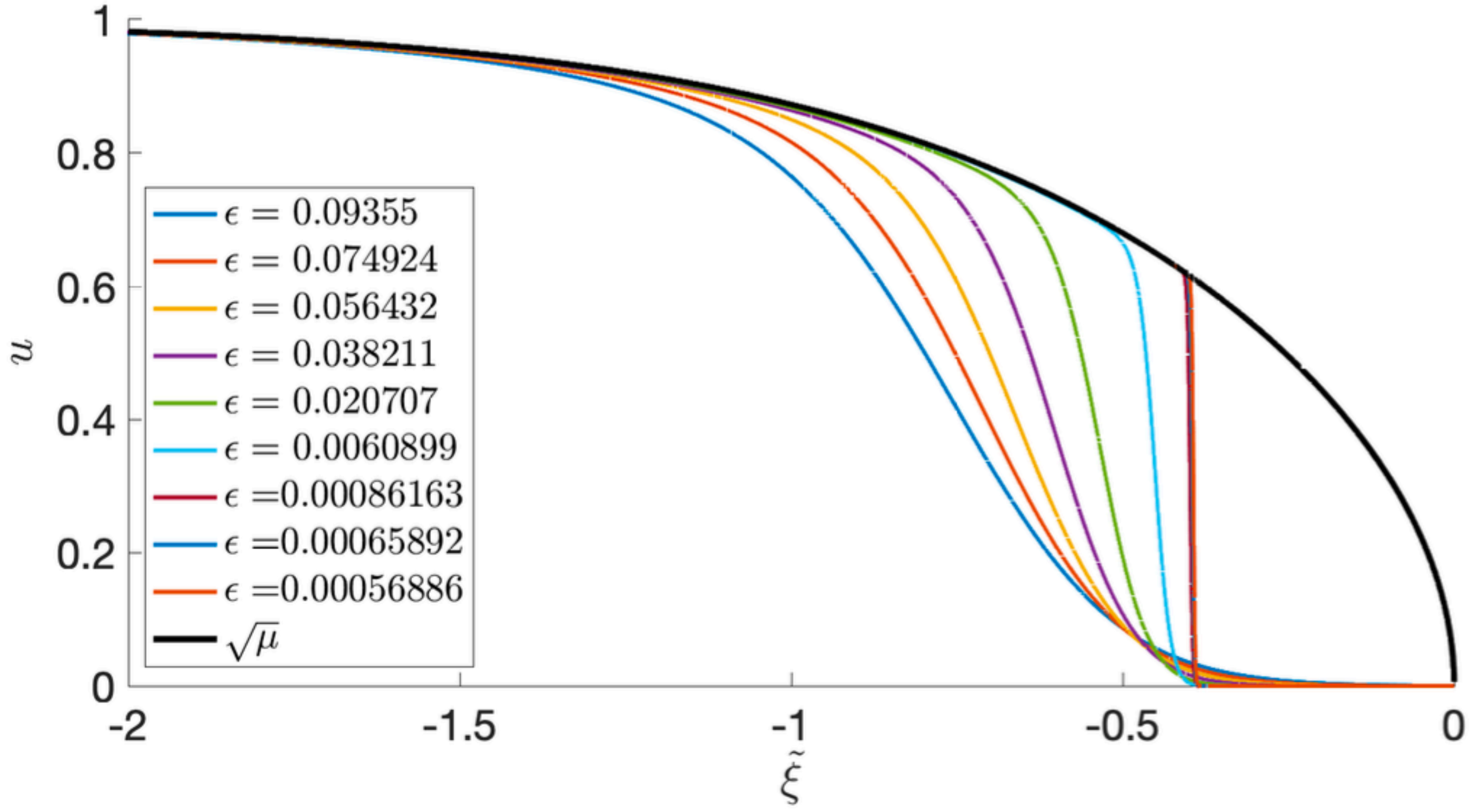


Secondary Delay!

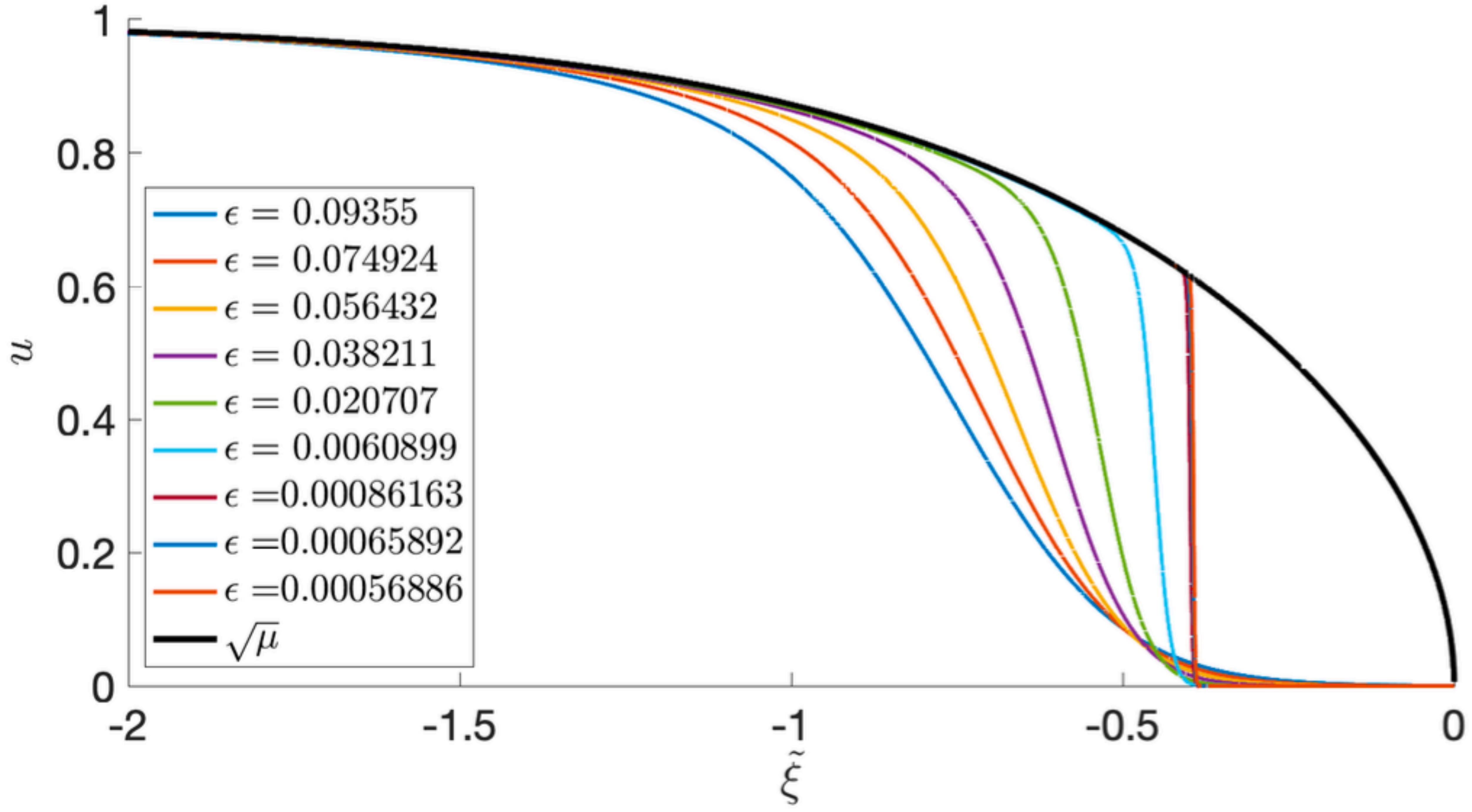


1
0.8
0.6
0.4
0.2
0
u
-2
-1.5
-1
-0.5
0
$\tilde{\xi}$
$\epsilon = 0.09355$
$\epsilon = 0.074924$
$\epsilon = 0.056432$
$\epsilon = 0.038211$
$\epsilon = 0.020707$
$\epsilon = 0.0060899$
$\epsilon = 0.00086163$
$\epsilon = 0.00065892$
$\epsilon = 0.00056886$
$\sqrt{\mu}$

Theorem: *Existence of transverse front solution for ϵ sufficiently small, $\boldsymbol{u}(\xi)$ positive, monotone, with front interface location given by*

$$\mu_{\mathrm{fr}} = \frac{c^2}{4} + \Omega_0 \left(1 - \frac{c^4}{16}\right)^{\frac{2}{3}} \epsilon^{\frac{2}{3}} + \mathcal{O}(\epsilon \log(\epsilon))$$

$\boldsymbol{\Omega_0}$ —> *Smallest positive root of* $J_{-1/3}(2z^{3/2}/3) + J_{1/3}(2z^{3/2}/3)$, J_n -> *Bessel Function of 1st kind*

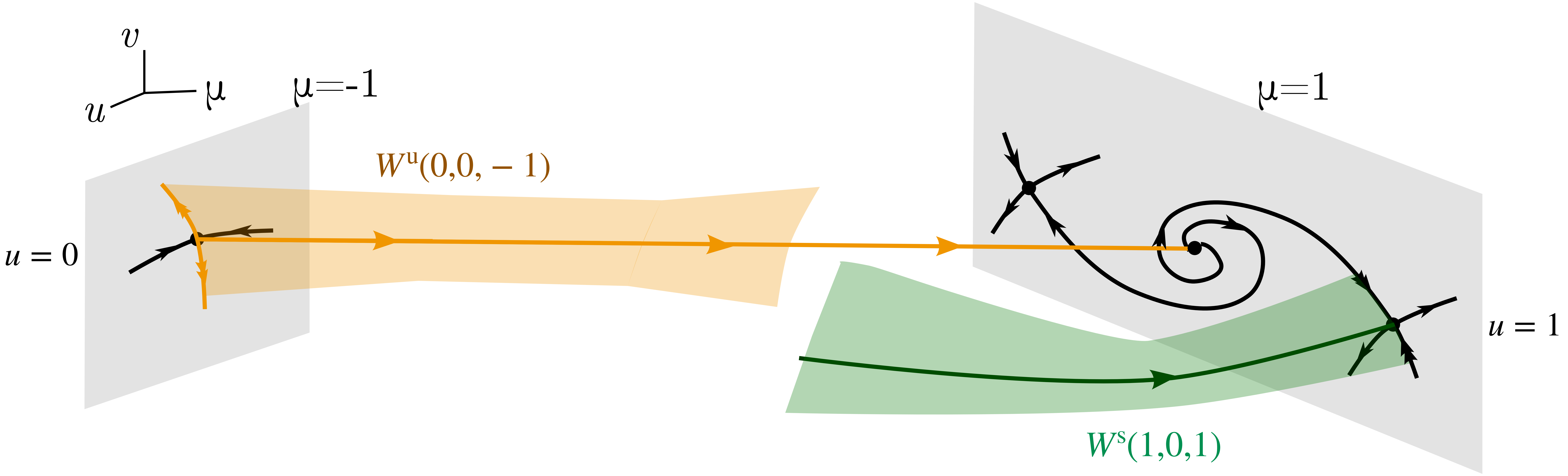


Heteroclinic intersection: $W^{u}(0,0,-1) \cap W^{s}(1,0,1)$

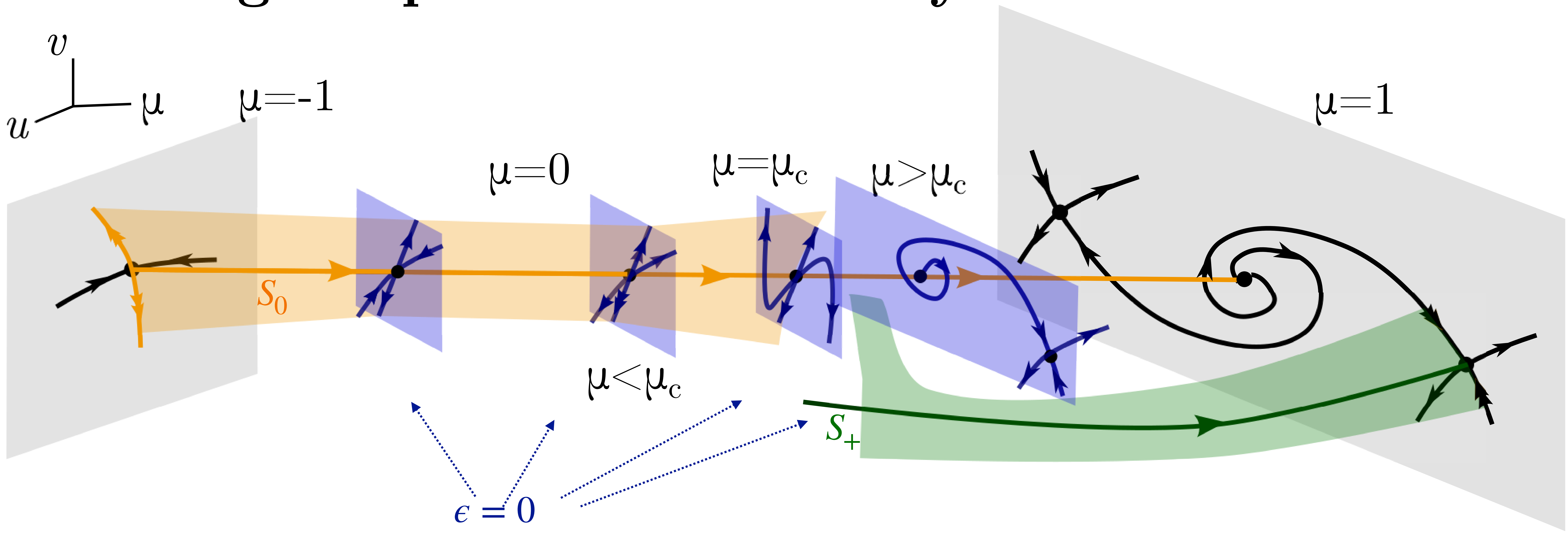
$$0 = u_{\xi\xi} + cu_{\xi} + \mu u - u^3,$$
$$\mu_{\xi} = -\epsilon(1-\mu^2)$$

$\zeta = -\xi$ →

$$u_{\zeta} = v$$
$$v_{\zeta} = cv - \mu u + u^3$$
$$\mu_{\zeta} = \epsilon(1-\mu^2)\,.$$



Geometric singular perturbation theory



Track W^u_ϵ and W^u_ϵ by studying $\epsilon = 0$ phase portraits:

- $W^u_0 = \cup_\mu W^{uu}_0$ for $\mu < 0$ when $S_0 = \{(0{,}0{,}\mu), \mu < 0\}$ is normally hyperbolic
- $W^s_0 = \cup_\mu W^{ss}_0$, for $\mu > 0$ when $S_+ = \{(\sqrt{\mu}{,}0{,}\mu), \mu > 0\}$ is normally hyperbolic

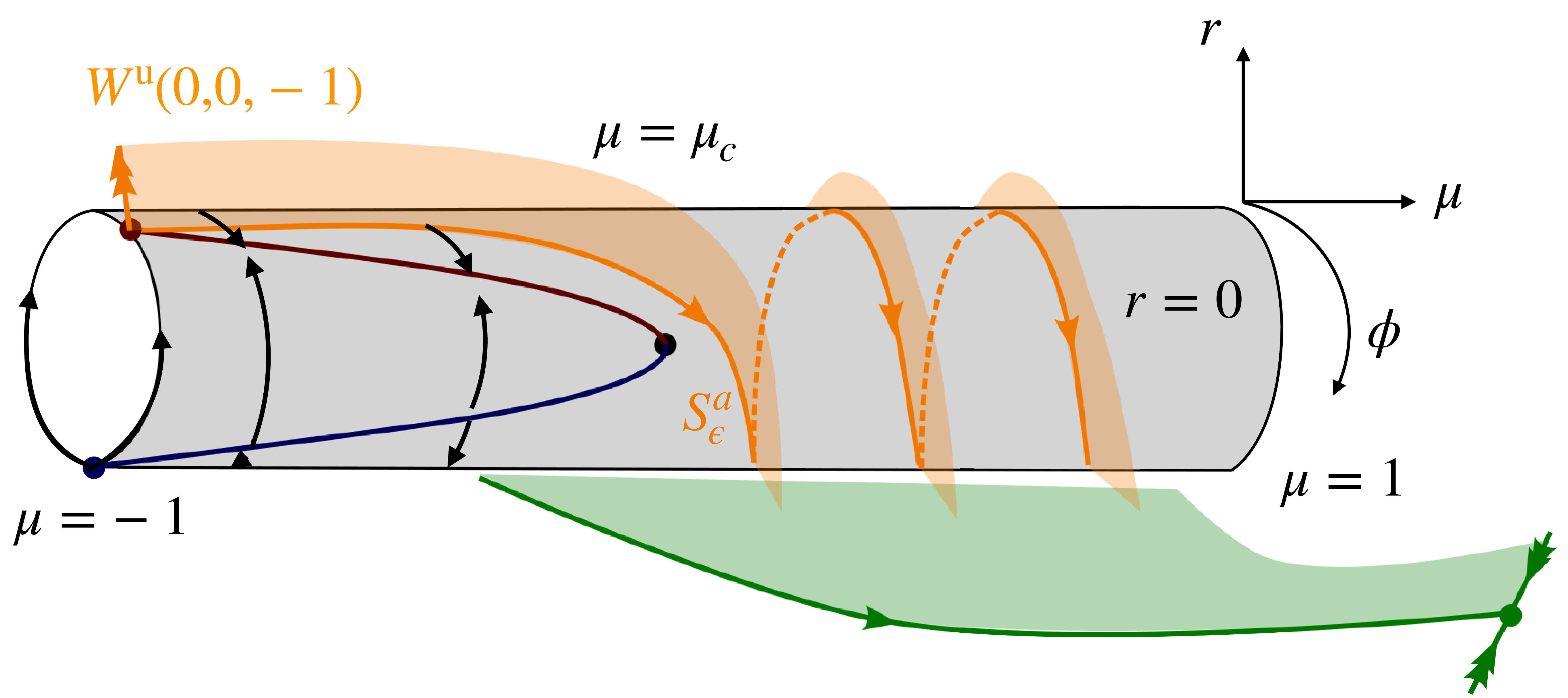
Fenichel theory $\Longrightarrow$ S_0 and S_+ persist as slow invariant manifolds for $0 < \epsilon \ll 1$ with strong foliations creating W^u_ϵ and W^u_ϵ

Difficulty: Tracking invariant manifolds through non-normally hyperbolic point at (0,0,0)

Desingularization: Blow-up of line $(u, v) = 0$, $\mu \in [-1,1]$

$$u_\zeta = v$$
$$v_\zeta = cv - \mu u + u^3$$
$$\mu_\zeta = \epsilon(1 - \mu^2).$$

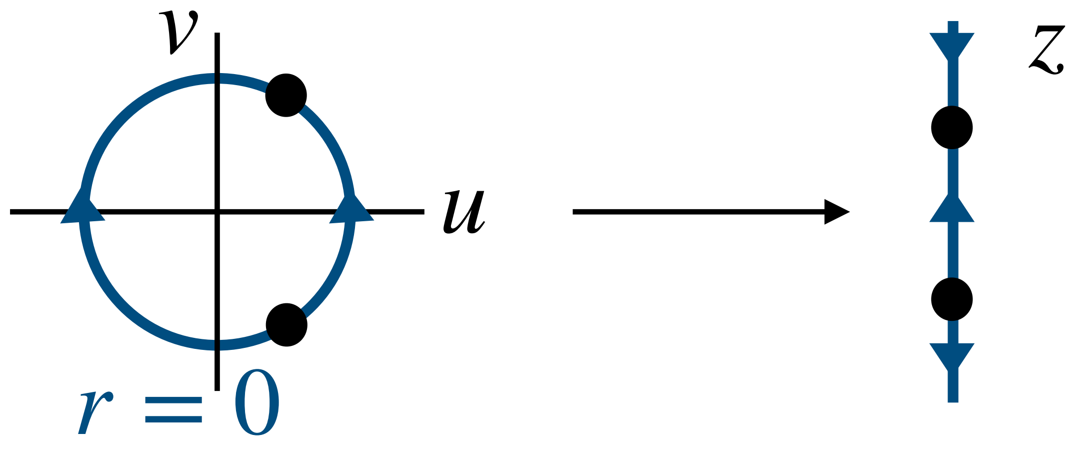
$$u = r\cos\phi, \quad v = r\sin\phi, \qquad (r, \phi) \in \mathbb{R}_+ \times [0,2\pi)$$



- $r = 0$ dynamics give flow on 1-Grassmanian, induced by linearized dynamics @ $(u, v) = (0,0)$
- $\epsilon = 0$: fold of equilibria curves S^a_0, S^r_0 at $\mu = c^2/4$, corresponds to real eigenspaces (in u, v dynamics) colliding in Jordan block yielding oscillatory dynamics
- $0 < \epsilon \ll 1$: Geometric singular perturbation theory [Fenichel] $\Longrightarrow$ slow-manifolds S^a_ϵ persists,

Projectivized Flow

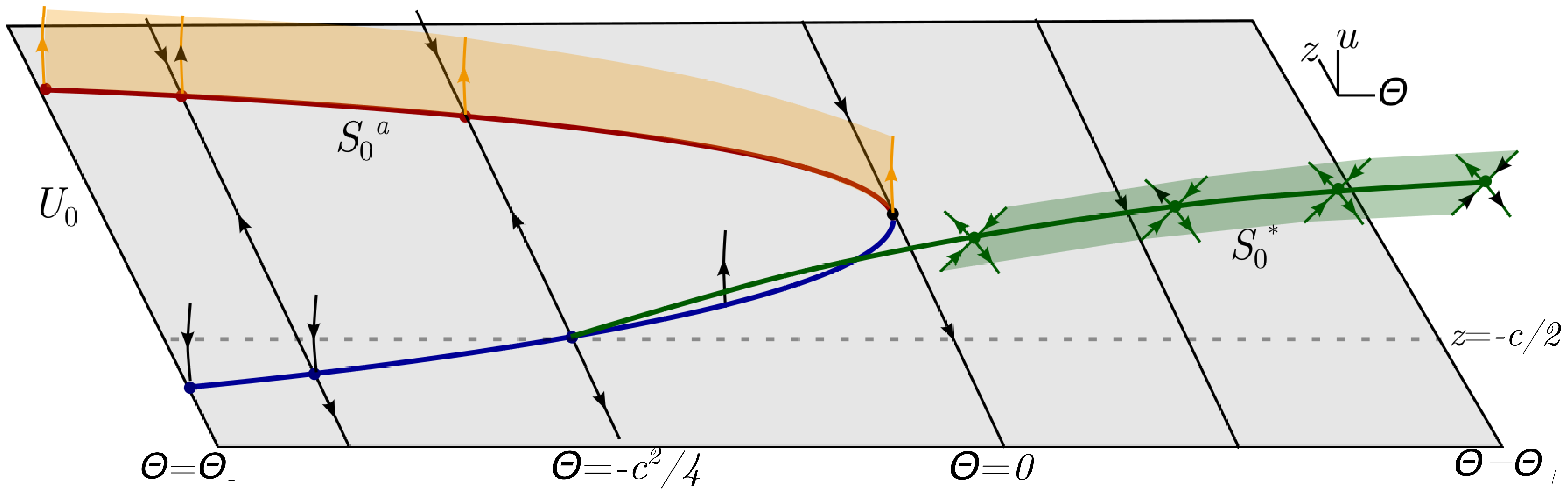
Projectivized coordinate chart: $z = v/u + c/2,\ u,\ \theta = \mu - c^2/4$



$$z_\zeta = -z^2 - \theta + u^2,$$
$$u_\zeta = (z + c/2)u,$$
$$\theta_\zeta = \epsilon(1 - (\theta + c^2/4)^2)\,.$$

$U_0 := \{u = 0\}$ is normally hyperbolic invariant manifold

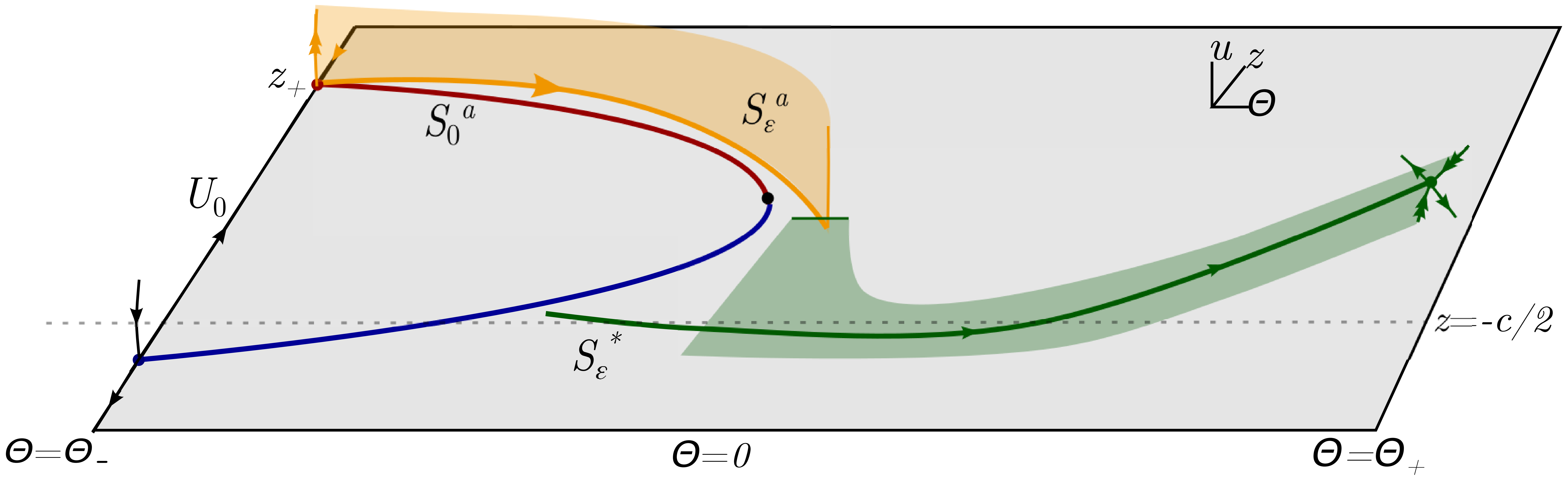
$\epsilon = 0$



$\mu = \mu_c = c^2/4$

Slow passage through a fold in U_0 dynamics:

$0 < \epsilon \ll 1$



Normal hyperbolicity and slow passage through a fold

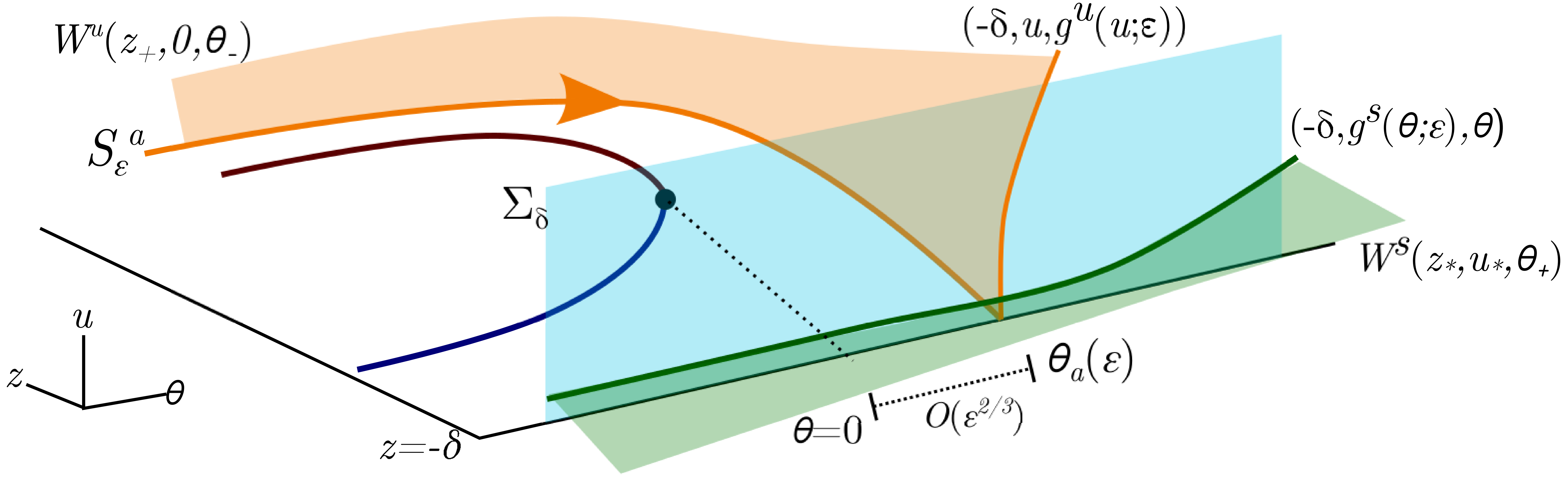
[Krupa & Szmolyan, 2001]:

Geometric blow-up of fold point in U_0 dynamics

—>

Bifurcation delay of slow manifold S^a_ϵ is $\mathcal{O}(\epsilon^{2/3})$

$$z_\zeta = -z^2 - \theta,$$
$$\theta_\zeta \approx \epsilon\,.$$



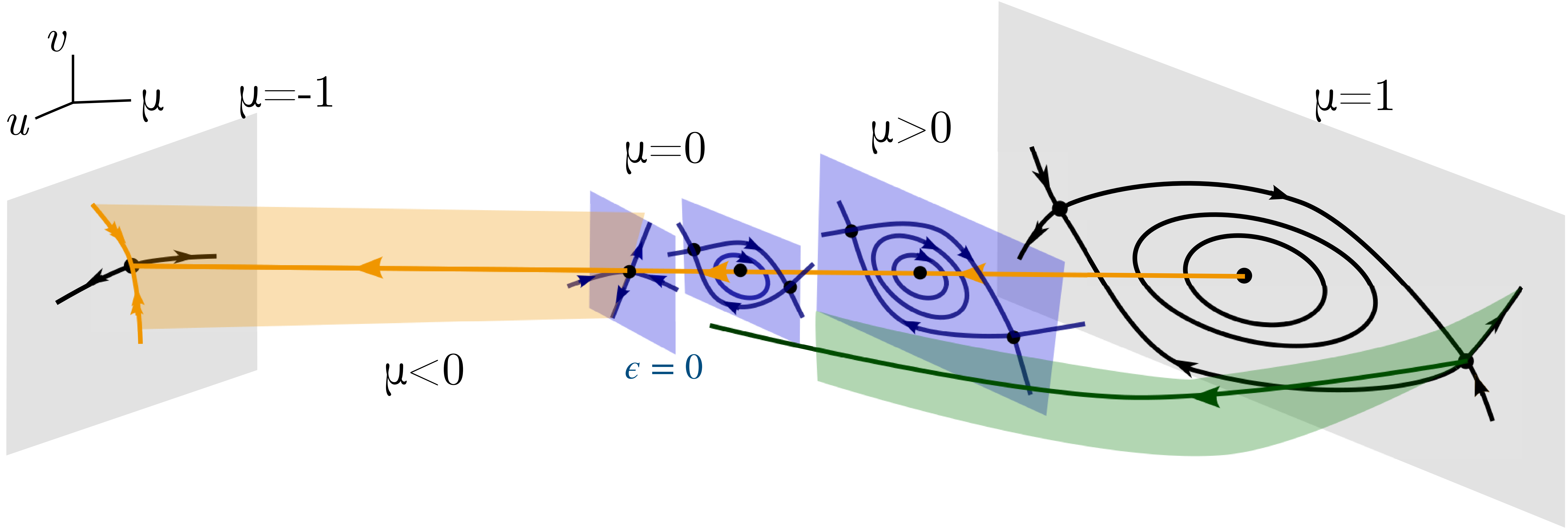
...use Fenichel theory, and normal hyperbolicity in the u-direction to locate heteroclinic intersection in neighborhood of U_0

Stationary case, $c = 0$: slow passage through a pitchfork

$$u_\xi = v$$

$$v_\xi = -\mu u + u^3$$

$$\mu_\xi = -\epsilon(1 - \mu^2)$$



See also [Haberman '80, Mareé '96, Krupa/Szmolyan '01] for related but different studies

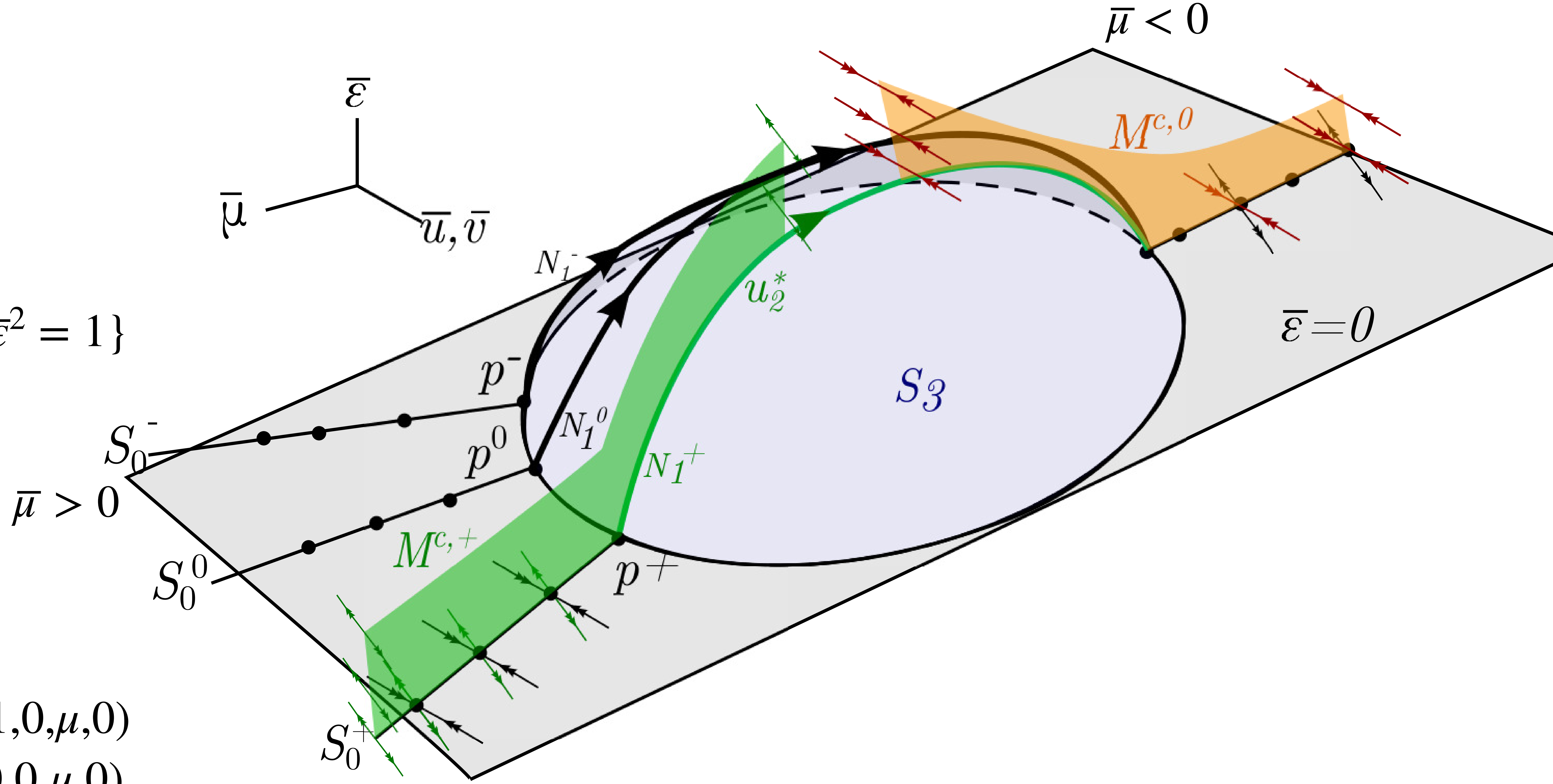
Stationary case, $c = 0$: slow passage through a pitchfork

$$u_\xi = v$$
$$v_\xi = -\mu u + u^3$$
$$\mu_\xi = -\epsilon(1-\mu^2)$$
$$\epsilon_\xi = 0.$$

Quasi-homogeneous blow-up: $u = r\bar{u}, \quad v = r^2\bar{v}, \quad \mu = r^2\bar{\mu}, \quad \epsilon = r^3\bar{\epsilon},$

Blows-up

$(0,0,0,0) \to S_3 = \{r = 0,\ \bar{u}^2 + \bar{v}^2 + \bar{\mu}^2 + \bar{\epsilon}^2 = 1\}$



S_0^+ - line of critical equilibria $(u, v, \mu, \epsilon) = (1,0,\mu,0)$

S_0^0 - line of critical equilibria $(u, v, \mu, \epsilon) = (0,0,\mu,0)$

... track evolution of invariant manifolds using coordinate charts

$K_1 \sim \bar{\mu} = 1$

$K_2 \sim \bar{\epsilon} = 1$

$K_3 \sim \bar{\mu} = -1$

Crucial part: rescaling chart $K_2 \sim \bar{\epsilon} = 1$

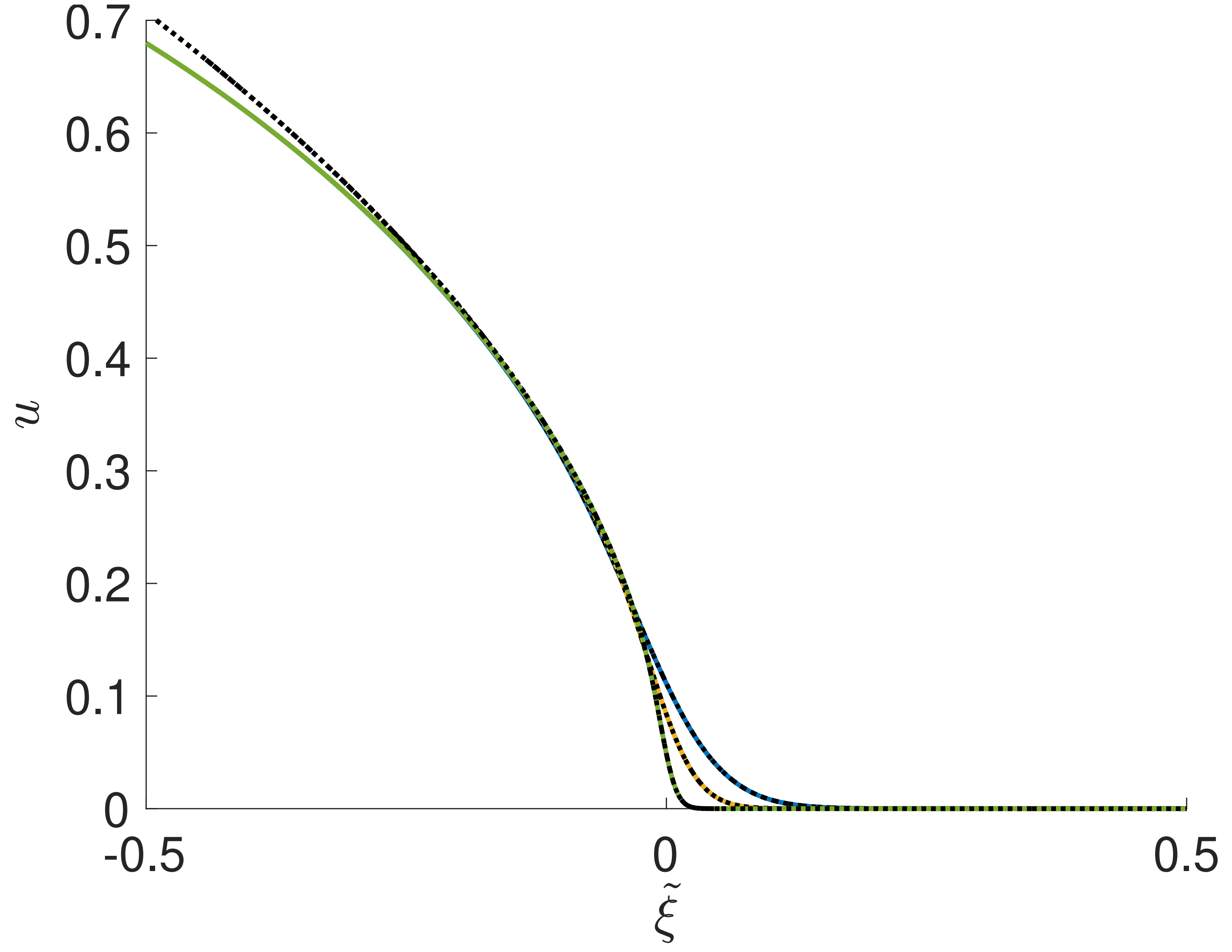
$u = r_2 u_2, v = r_2^2 v_2, \mu = r_2^2 \mu_2, \epsilon = r_2^3$, and time re-scaling:

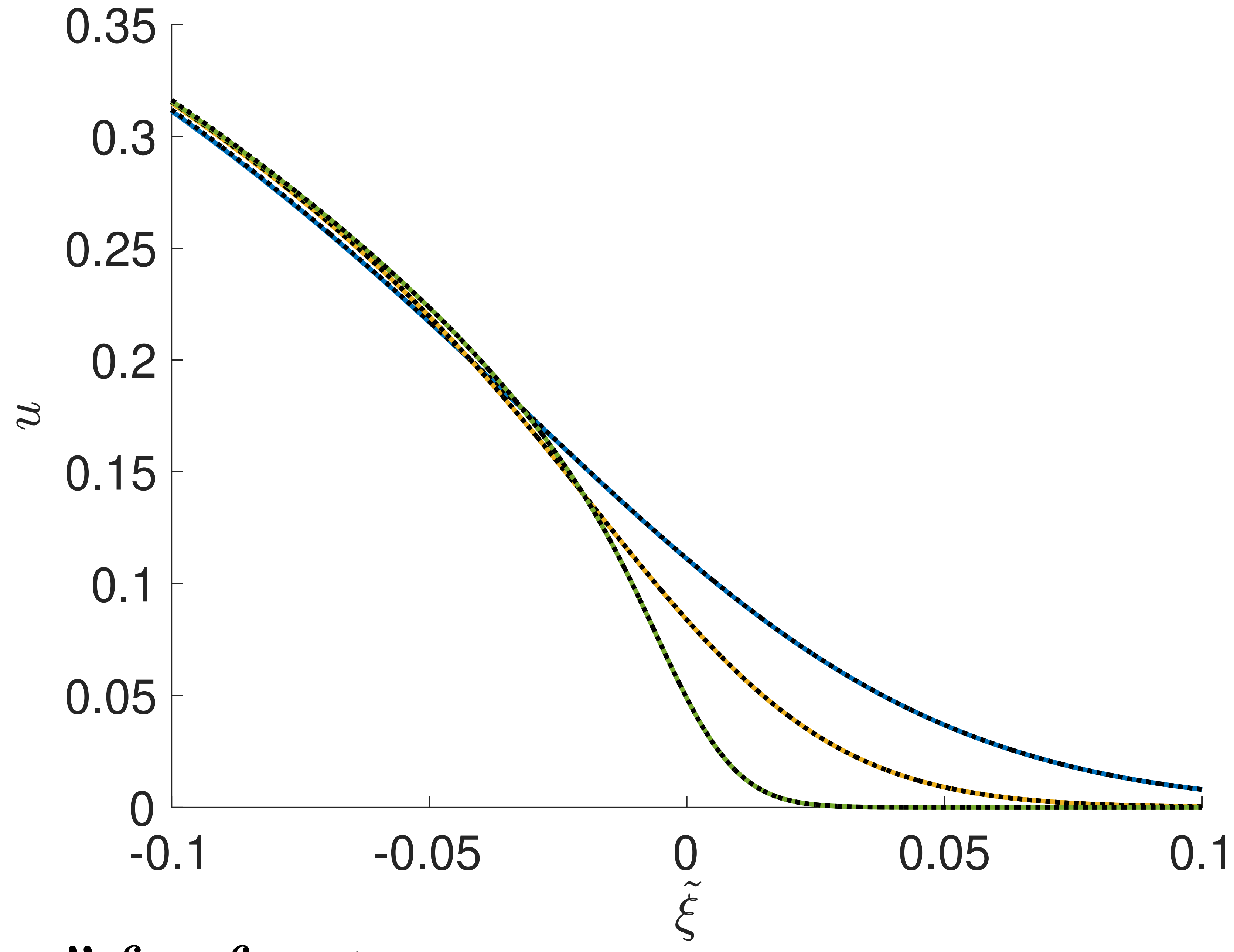
$$\begin{aligned} u_2' &= v_2 \\ v_2' &= -\mu_2 u_2 + u_2^3 \\ \mu_2' &= -1 + r_2^4 \mu_2^2 \\ r_2' &= 0. \end{aligned} \quad \xrightarrow[\text{Restrict to sphere}]{r_2 = 0} \quad \begin{aligned} u_2' &= v_2 \\ v_2' &= -\mu_2 u_2 + u_2^3 \\ \mu_2' &= -1 \\ r_2' &= 0. \end{aligned} \quad \xrightarrow{u_2 = \sqrt{2}\tilde{u}_2} \quad \tilde{u}_2'' = \xi_2 \tilde{u}_2 + 2\tilde{u}_2^3.$$

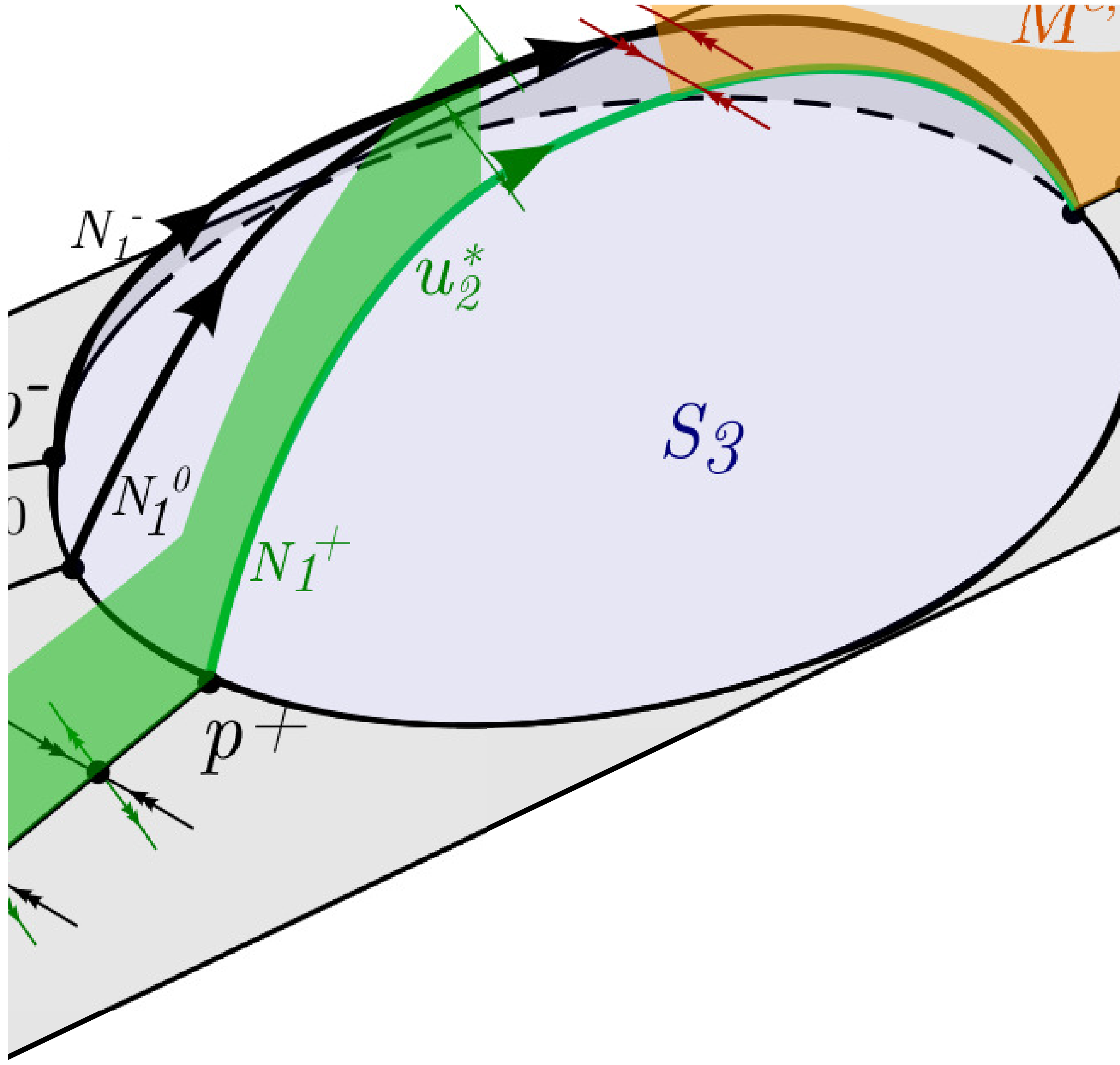
Painlevé's 2nd Equation!!!

[Hastings & McLeod 1980] —> Exists a unique connecting orbit $\tilde{u}_2^*$ with

$$\tilde{u}_2^*(\xi_2) \sim \sqrt{-\xi_2/2},\ \xi_2 \to -\infty, \qquad \tilde{u}_2^*(\xi_2) \sim \mathrm{Ai}(\xi_2),\ \xi_2 \to +\infty,$$



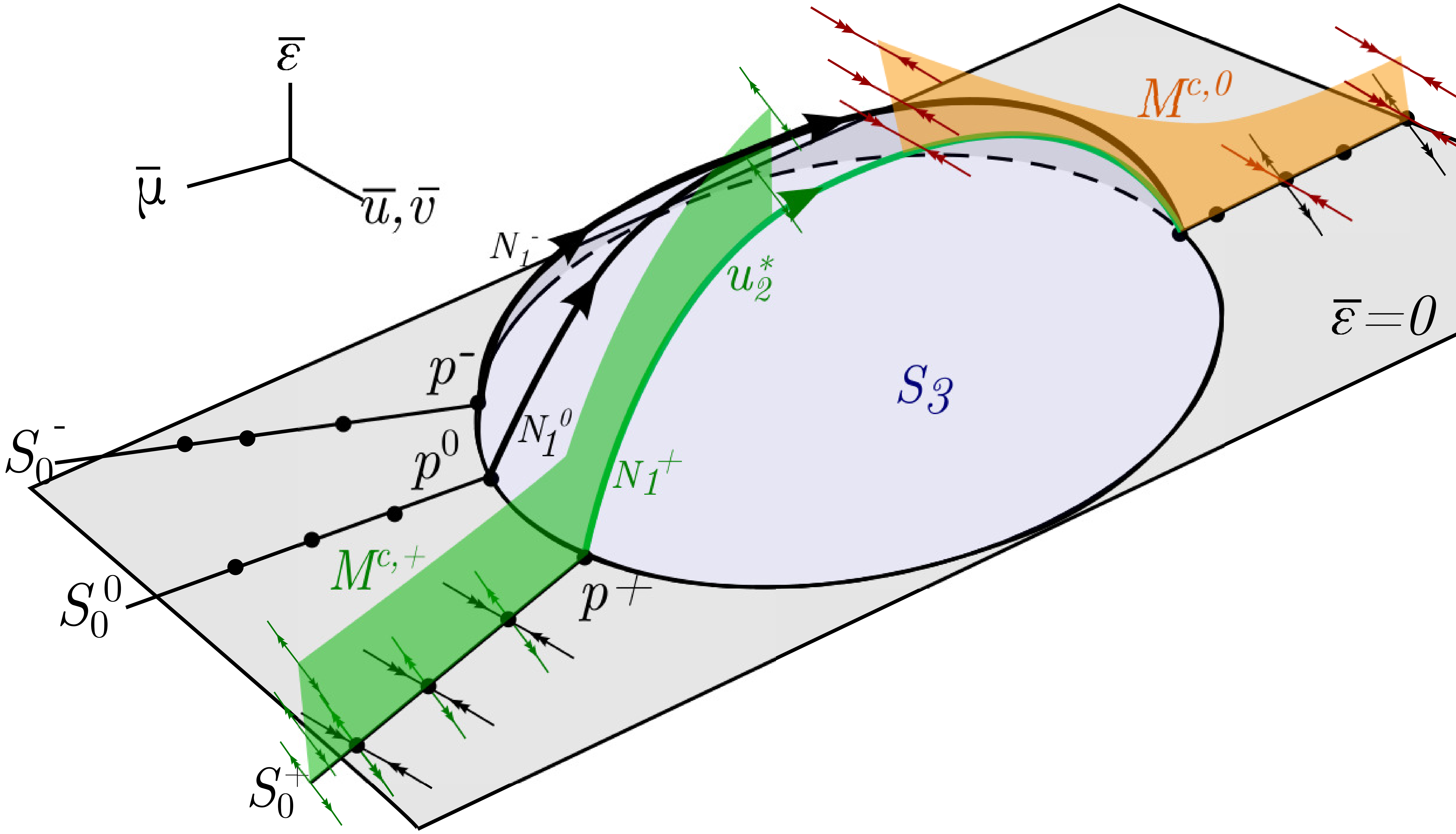




"Inner solution" for front

Rest of proof:

- Use monotonicity of the linearized operator $L_0 w = w'' - (\xi_2 + 3(u_2^*)^2)w$ to obtain transversality of Hastings-McLeod orbit u_2^* (new!)
- Use exponential trichotomies and inclination properties of K_1, K_3 charts to conclude transverse intersection of nearby center-stable and center-unstable manifolds



Theorem: *Existence of heteroclinic orbit for all* $\epsilon > 0$ *sufficiently small, with inner asymptotics*

$$u^*(\xi) = u_{HM}(\xi) + \mathcal{O}(\epsilon^{2/3}), \qquad |\xi| \le \rho\epsilon^{-1/3},$$

where $u_{HM}(\xi) = \sqrt{2}\epsilon^{1/3} w_{HM}(\epsilon^{1/3}\xi)$ *and* w_{HM} *is the unique Hastings-McLeod connecting solution*

Stability in one slide

Co-moving frame: $u_t = u_{\zeta\zeta} - cu_\zeta + \mu u - u^3 \xrightarrow{\text{Linearize}} \mathscr{L}_0 v = v_{\zeta\zeta} - cv_\zeta + (\mu - 3(u^*)^2)v$

Spectral Stability

- Stable essential spectrum by study of asymptotic states

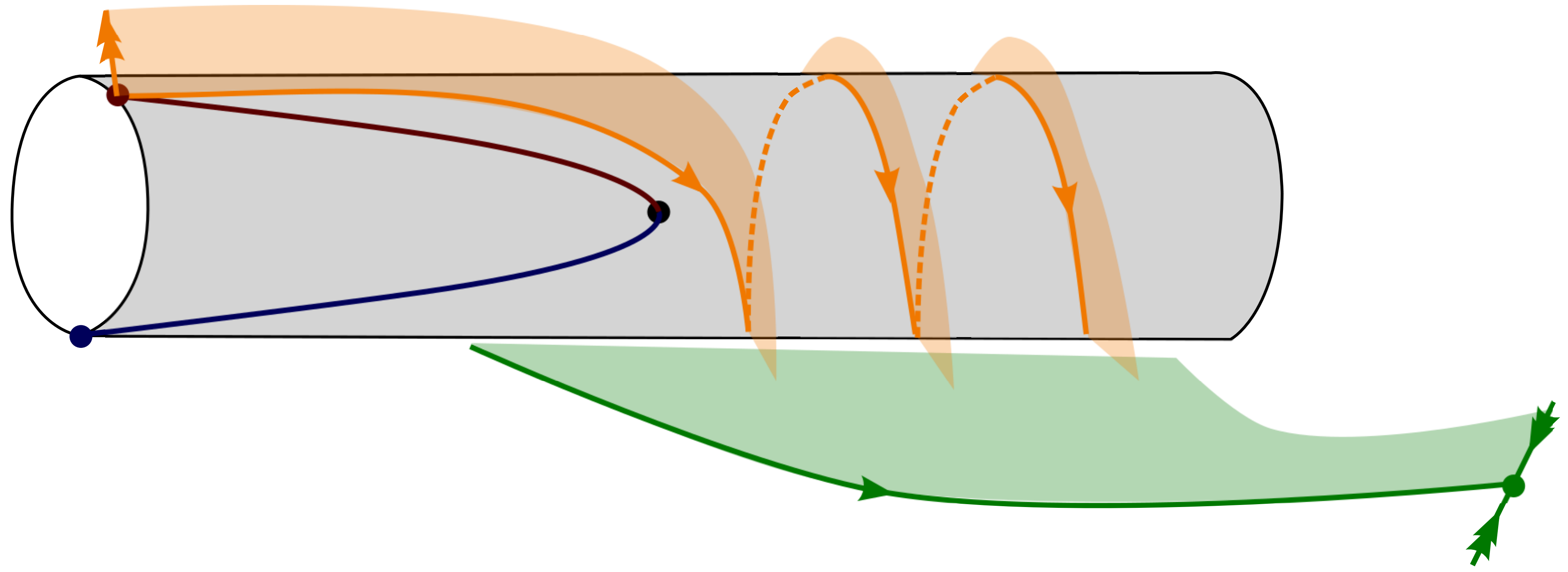
Point-spectrum:

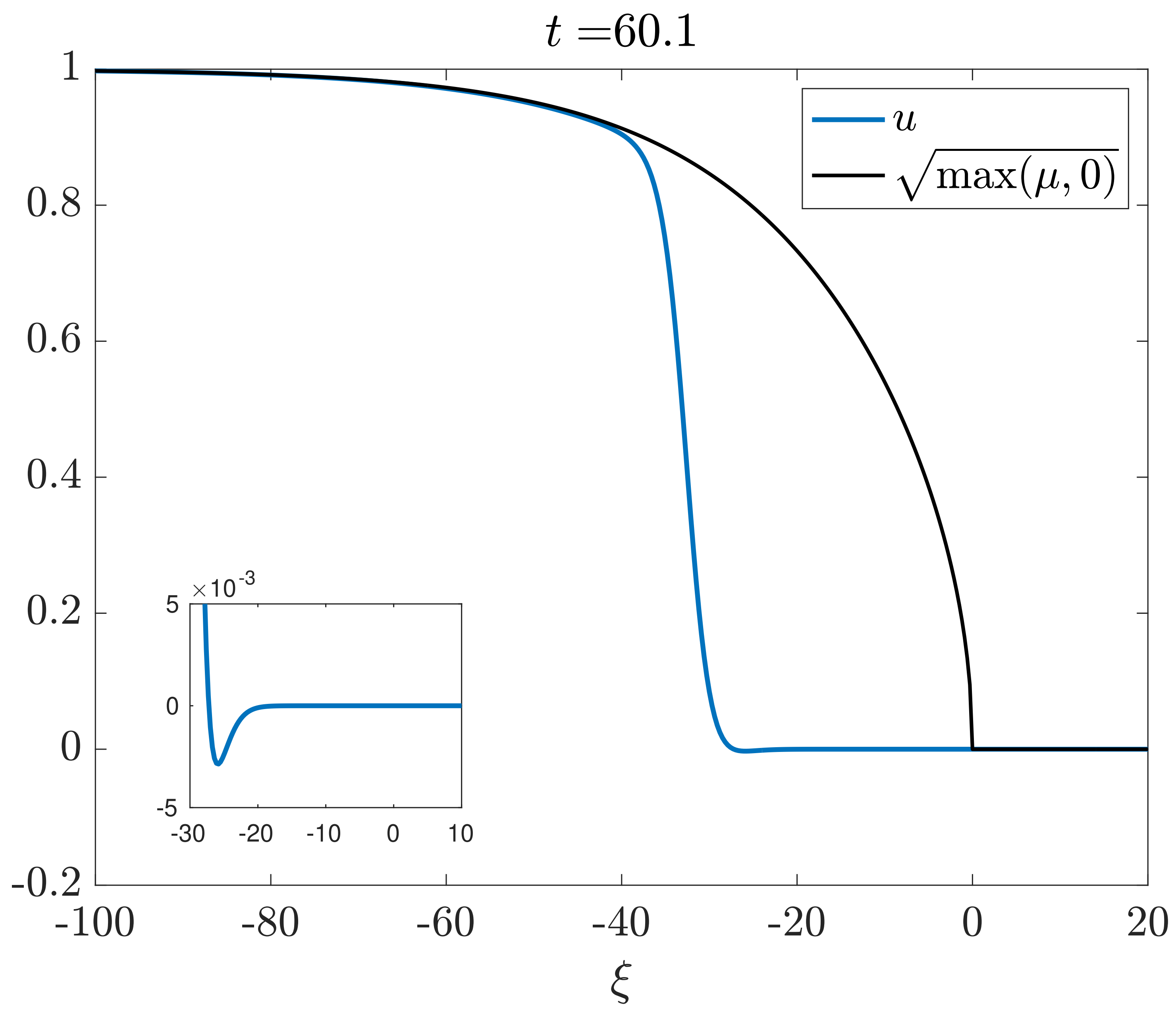
- Conjugate $\mathscr{L}_c v := (e^{-c\zeta/2}\mathscr{L}_0 e^{c\zeta/2})v = v_{\zeta\zeta} + (\mu - \frac{c^2}{4} - 3(u^*)^2)v,$
- Apply Sturm-Liouville/maximal eigenvalue argument (with strictly positive eigenfunction) to show all spectrum has $\lambda < 0$
- Note: no translational eigenvalue at zero!

Standard theory then allows one to conclude linear and nonlinear stability of front u^* in co-moving frame

Remark: Slow absolute spectrum work by [RG, de Rijk, 2022] or [Carter, Rademacher,Sandstede] should yield spectral stability in systems without monotonicity properties

Unstable non-monotonic fronts, $0 < c < 2$





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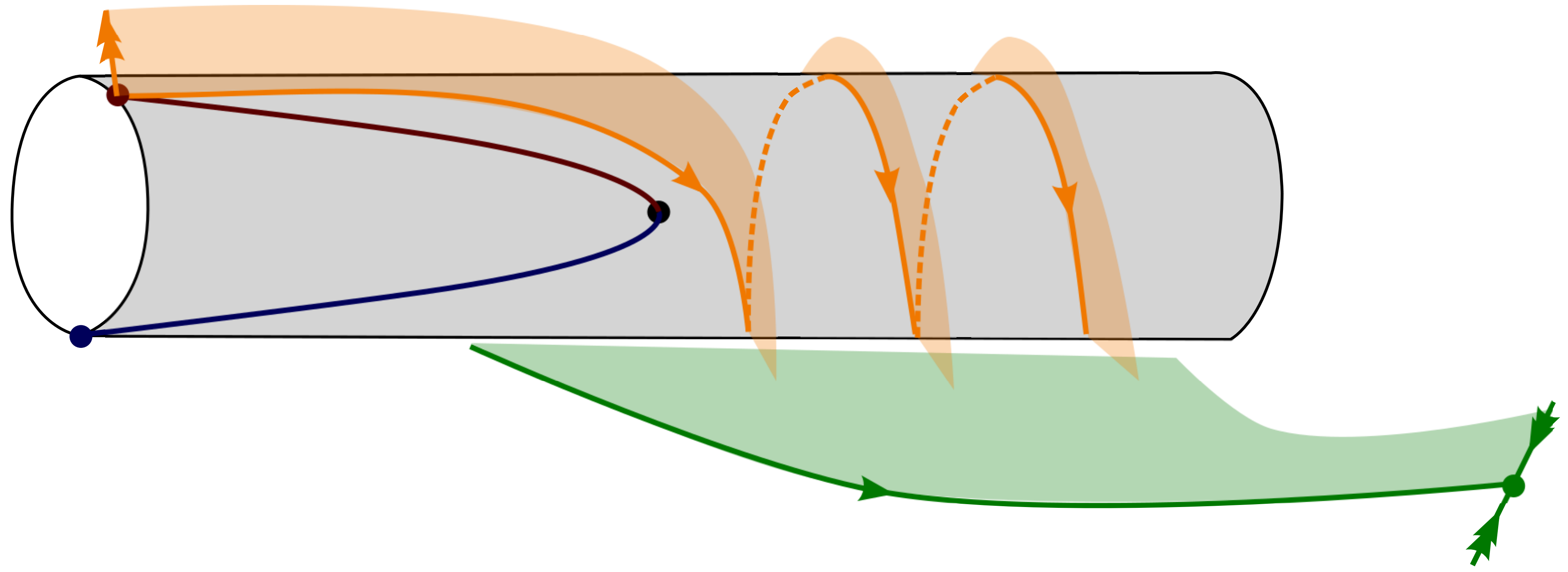
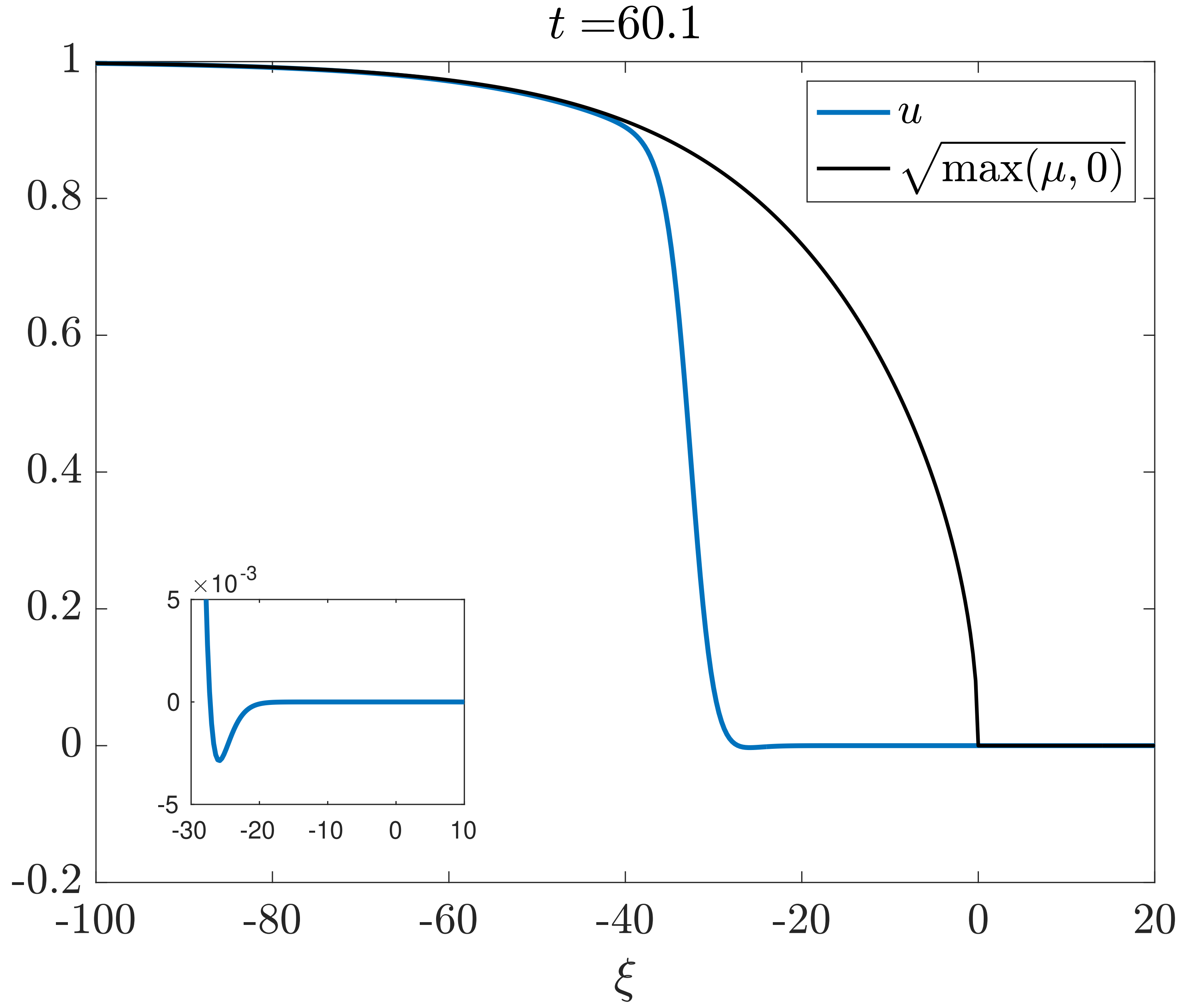
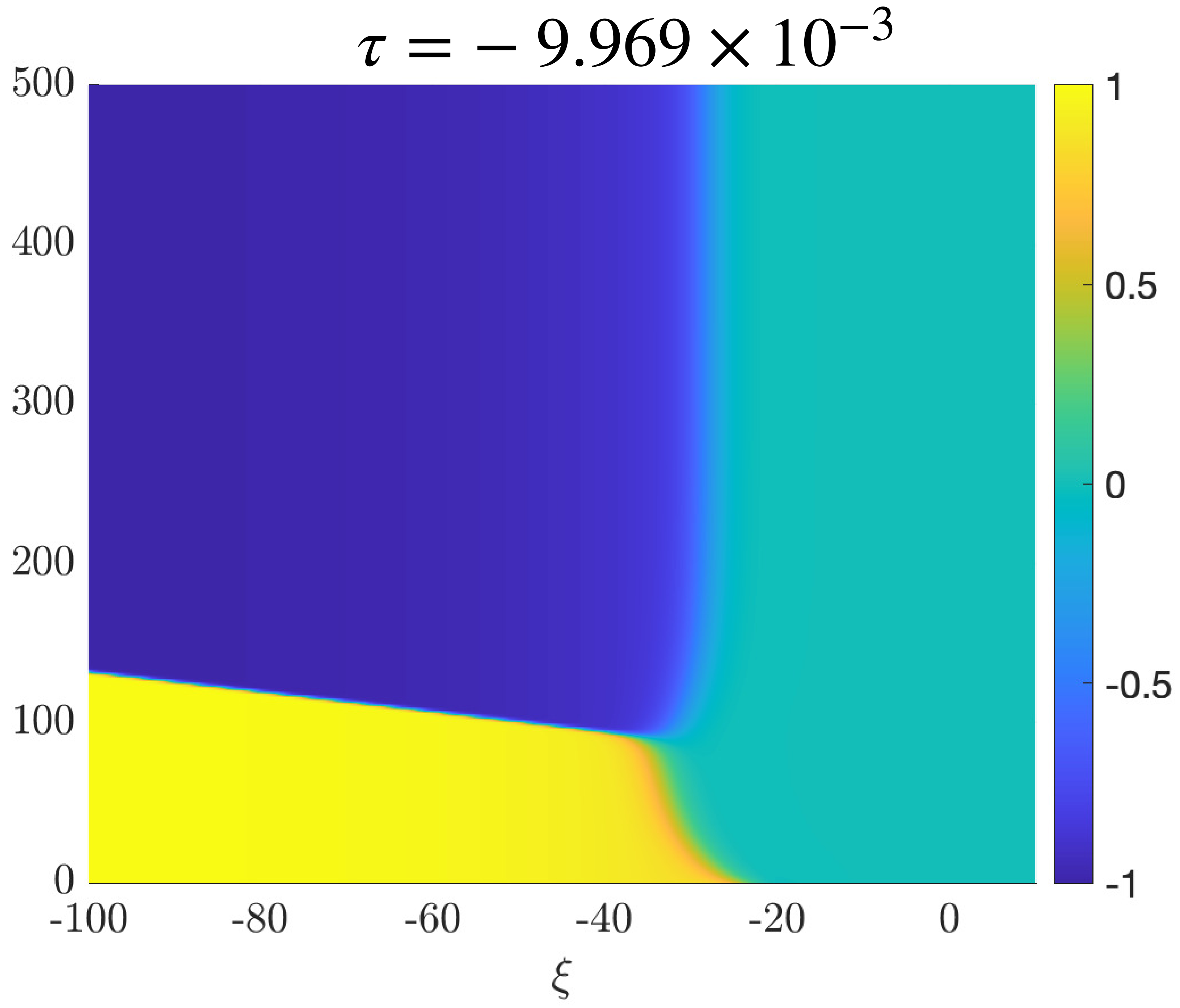
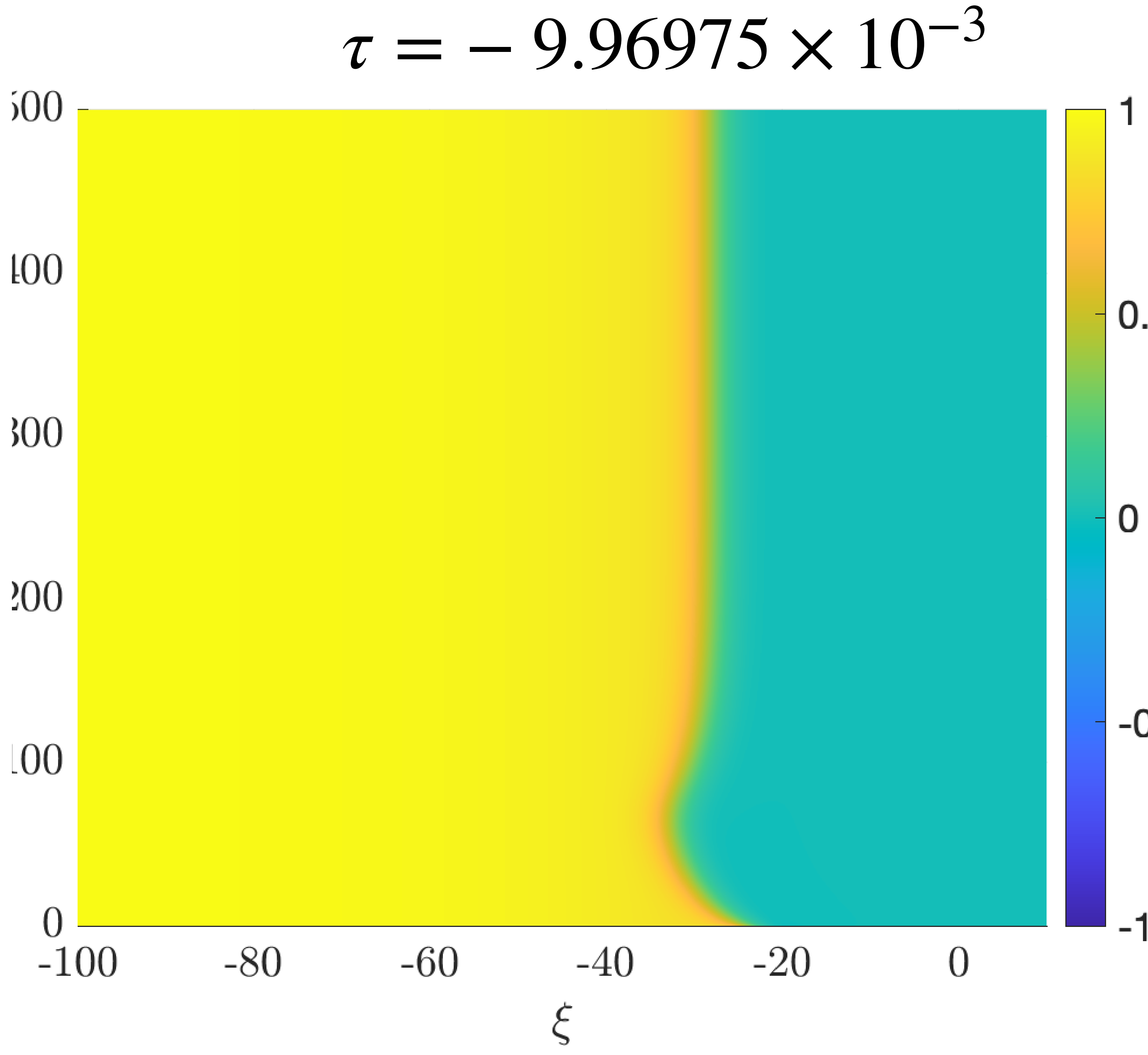




Initial Data with small negative interval

$$u^\tau(\xi,0) = \begin{cases} 1, & \xi < (c^2/4 - 0.1)/\epsilon \\ \tau, & (c^2/4 - 0.1)/\epsilon \le \xi \le (c^2/4 + 0.1)/\epsilon \\ 0, & \xi > (c^2/4 + 0.1)/\epsilon\,. \end{cases}$$





Gluing the two regimes: $c \sim 0, \qquad c = \mathcal{O}_\epsilon(1), \quad 0 < c < 2$

Study the regime $c \sim \epsilon^{1/3}$, here transition occurs near $\mu = 0$,

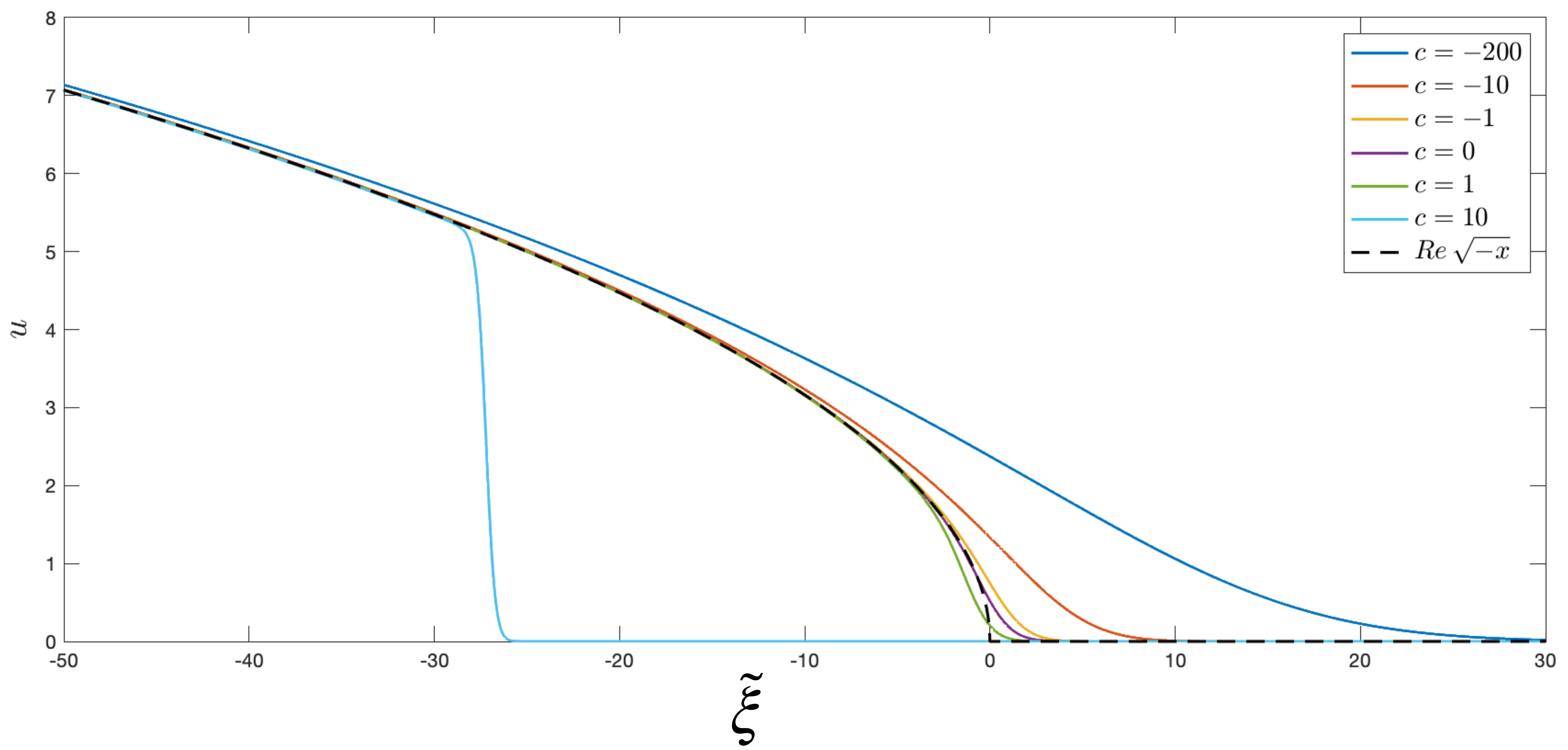
$\implies \mu = -\tanh(\epsilon\xi) \sim -\epsilon\xi$

Critical scaling: $\quad u = \epsilon^{1/3}\tilde{u}, \qquad \xi = \epsilon^{-1/3}\tilde{\xi}, \qquad c = \epsilon^{1/3}\tilde{c}, \qquad t = \epsilon^{-2/3}\tilde{t},$

Singular sphere eqn. $\implies$ Painlevé-II with drift! $\qquad 0 = \tilde{u}_{\tilde{\xi}\tilde{\xi}} + \tilde{c}\tilde{u}_{\tilde{\xi}} - \tilde{\xi}\tilde{u} - \tilde{u}^3 \qquad (*)$

Theorem: *For all $\tilde{c} \in \mathbb{R}$, there exists unique monotone decreasing front $\tilde{u}_*(\tilde{x};\tilde{c})$ of $(*)$ with*

$$\lim_{\tilde{\xi}\to-\infty}\left(\tilde{u}_*(\tilde{\xi};\tilde{c}) - \sqrt{-\tilde{\xi}}\right) = 0, \qquad \lim_{\tilde{\xi}\to\infty}\tilde{u}_*(\tilde{\xi};\tilde{c}) = 0.$$



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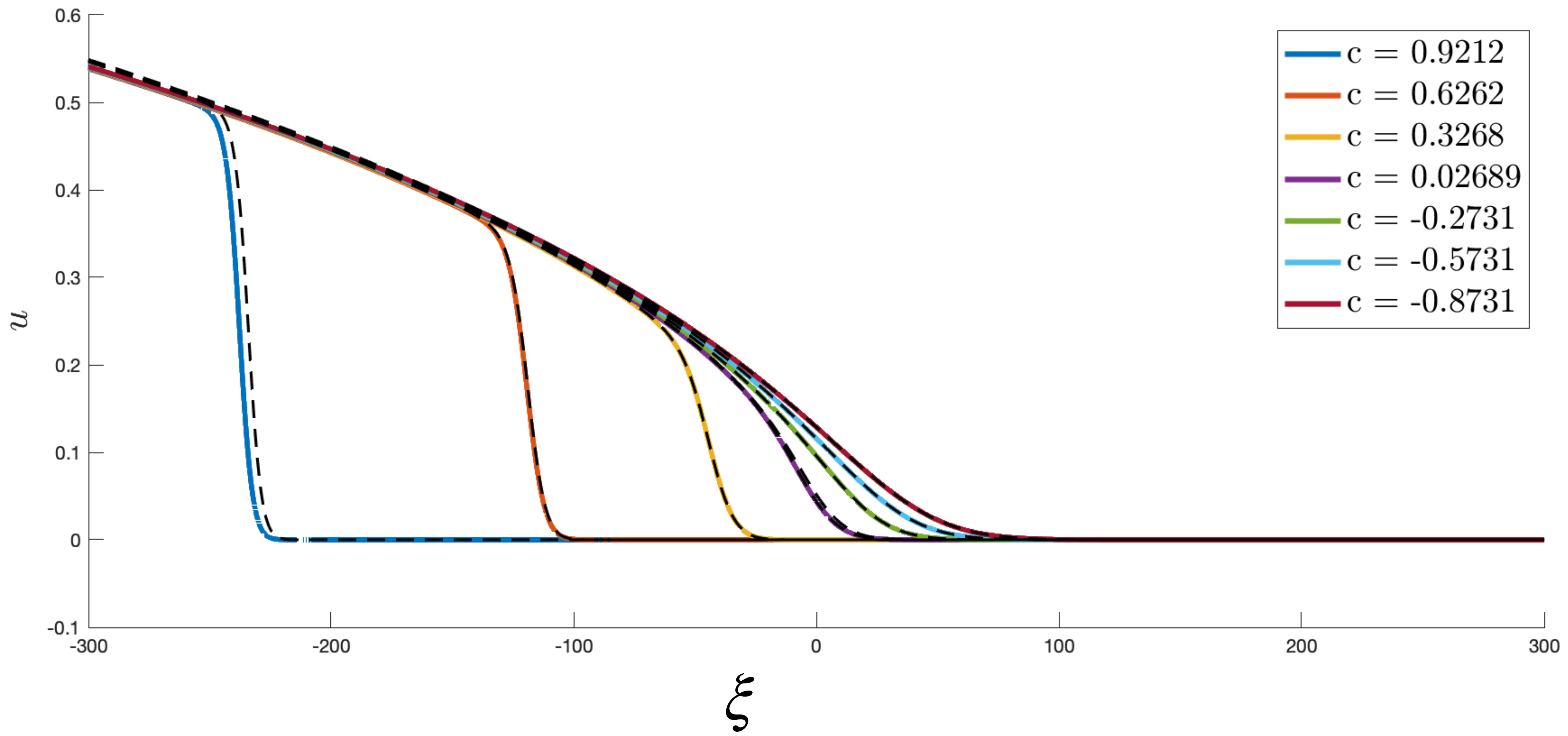
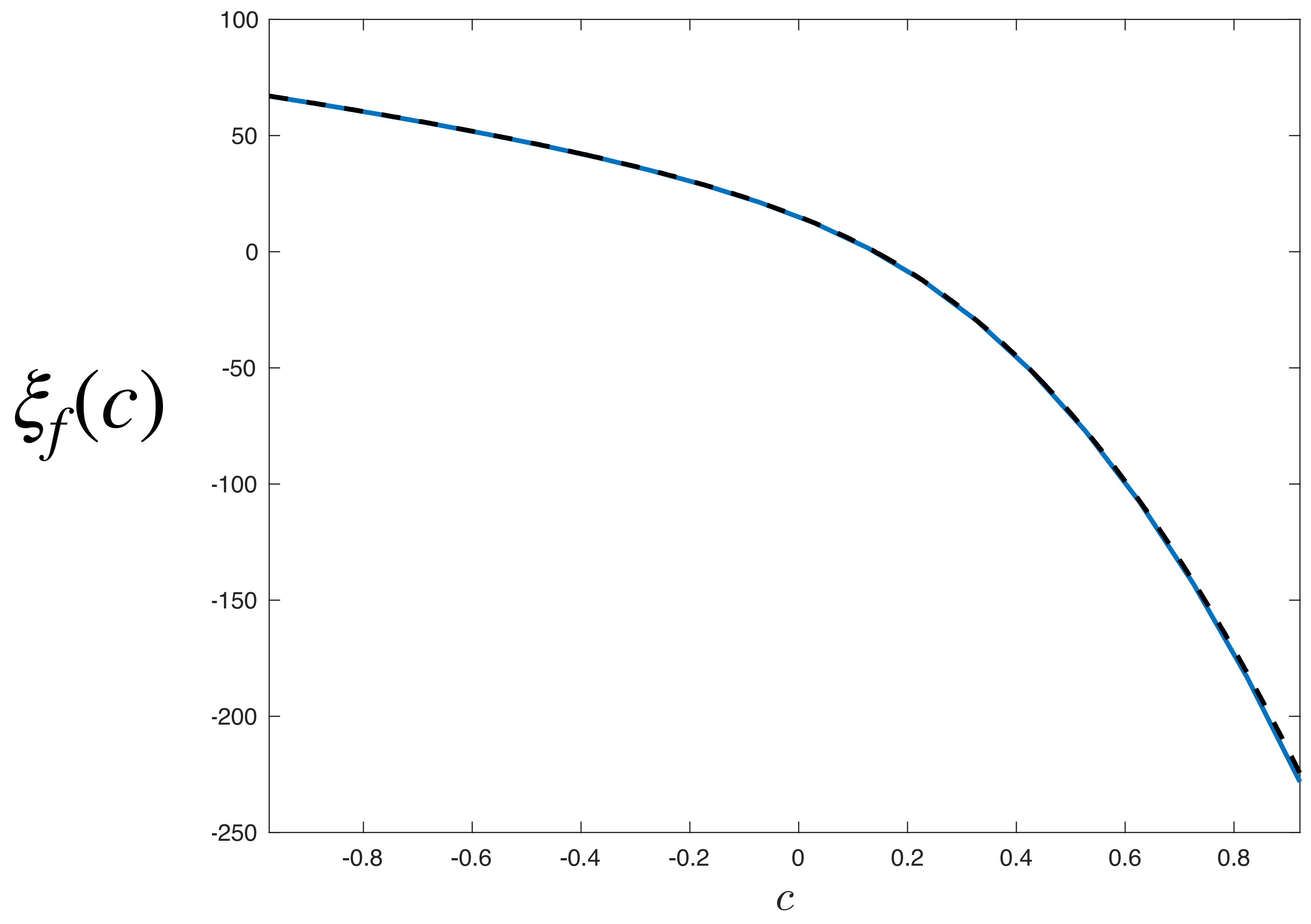
Allen-Cahn
P-II with drift





Outline of proof (in 1 slide)

Dynamics arguments (center manifold expansions, GSPT, blow-up....) :

(i) Exist front solution for each $\tilde{c} \ll -1$

(ii) Exist front solutions for each $\tilde{c} \gg 1$

Functional analysis:

(iii) Linearization $\mathscr{L}_{\tilde{c}} := \partial_{\tilde{\xi}\tilde{\xi}} + \tilde{c}\partial_{\tilde{\xi}} - (\tilde{\xi} + 3\tilde{u}_*^2)$ is a negative operator

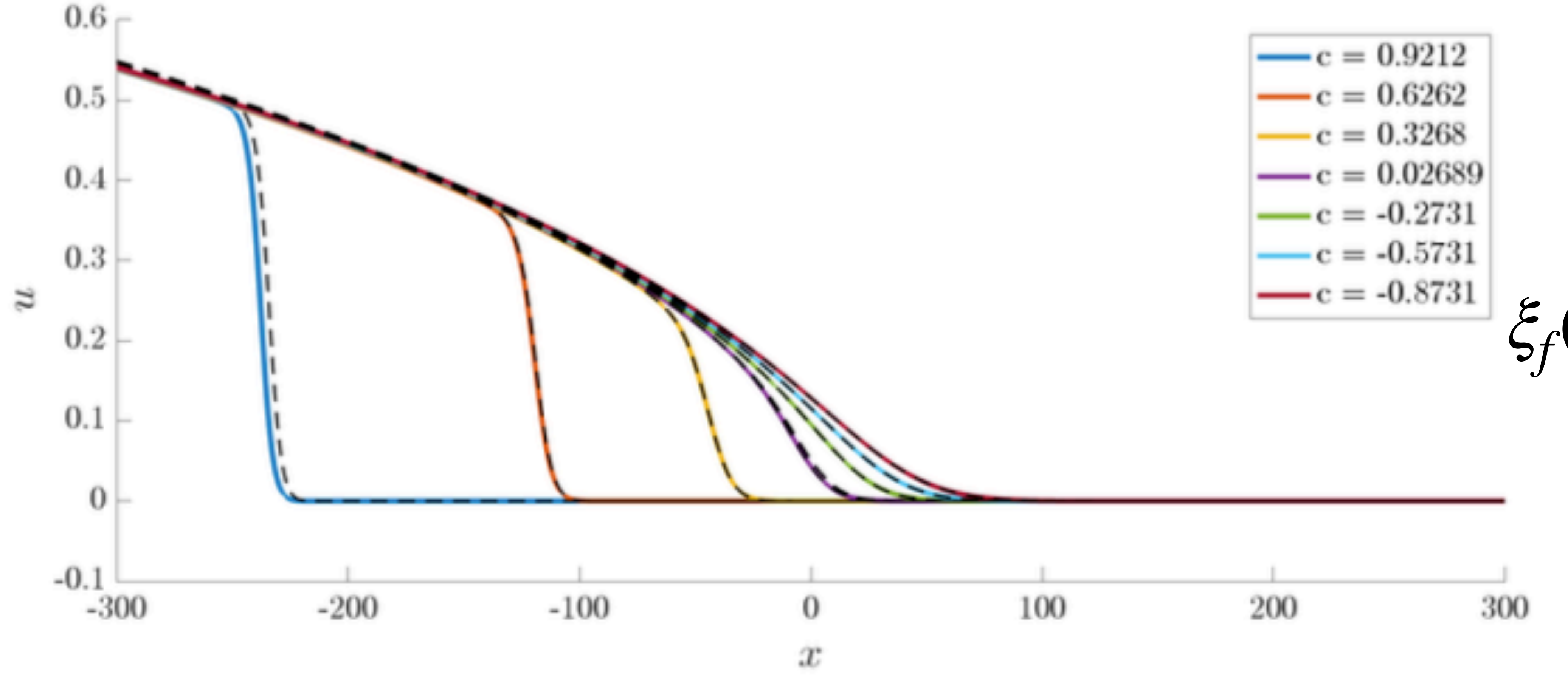
(iv) Implicit function theorem $\Longrightarrow$ set of monotone fronts is open

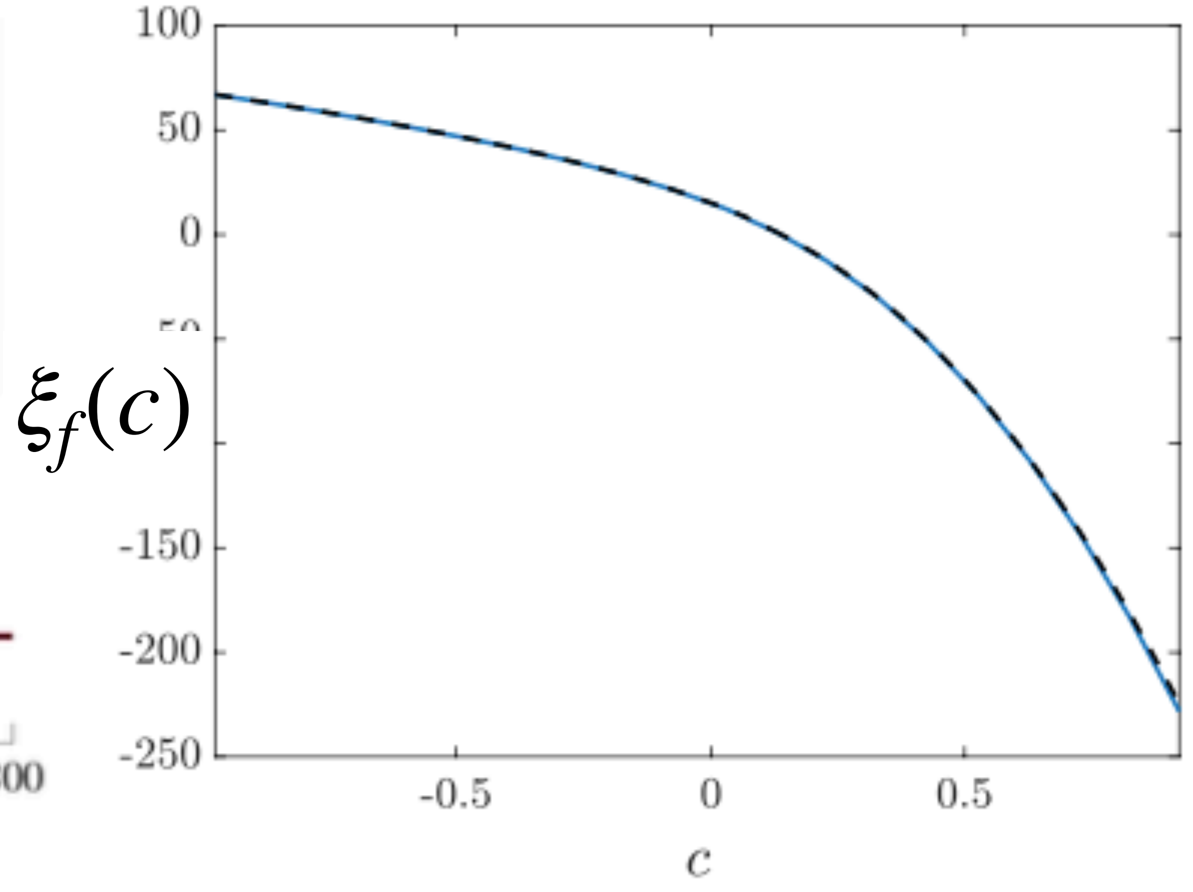
(v) Compactness argument $\Longrightarrow$ set of monotone fronts is closed

$\Longrightarrow$ existence for all $\tilde{c}$

<u>Remark</u>: To complete analysis of Allen-Cahn from (*), would need a global heteroclinic gluing argument similar to above, (*) does give accurate ``inner” solution!

- - - - - - P-II with drift inner solution





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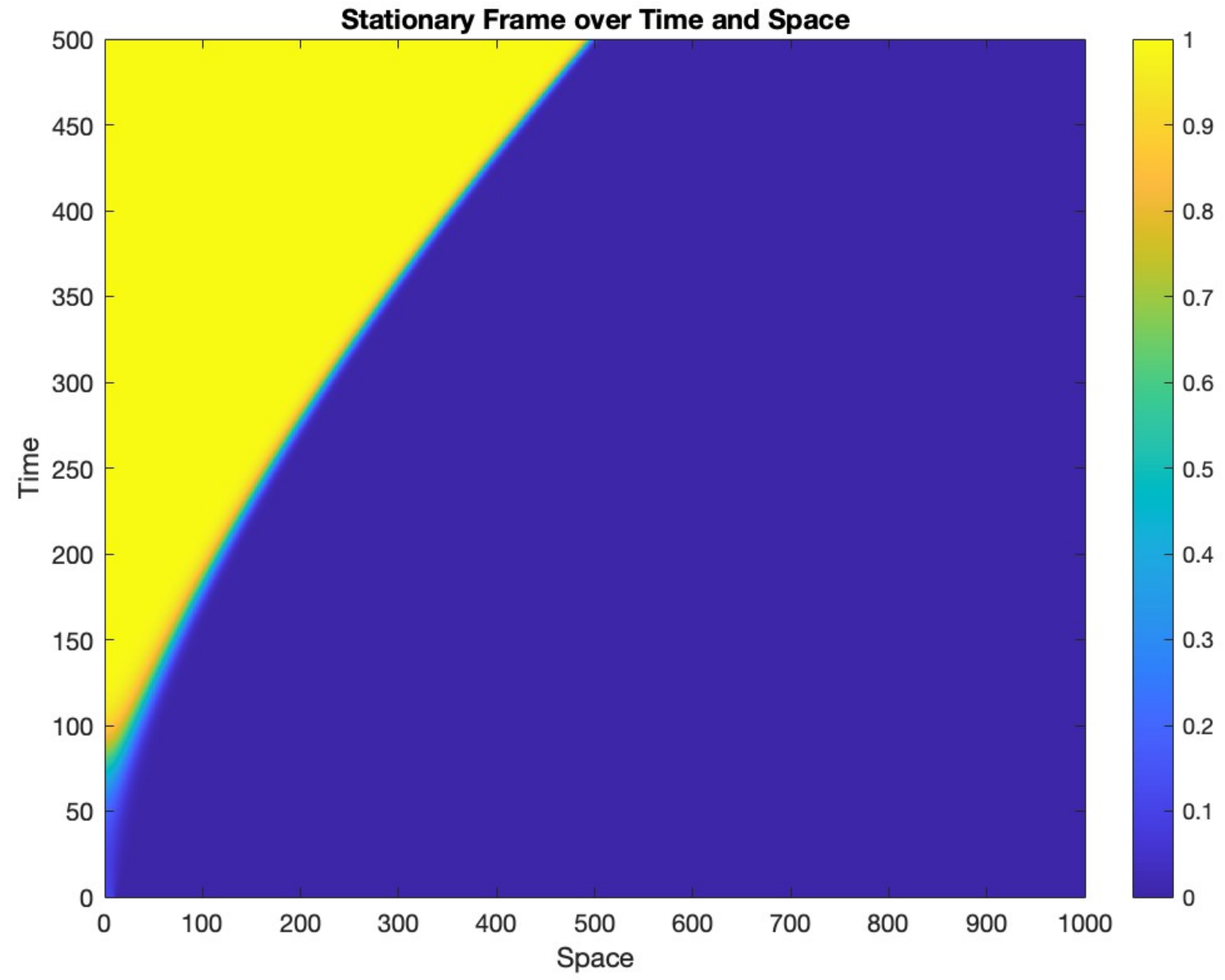
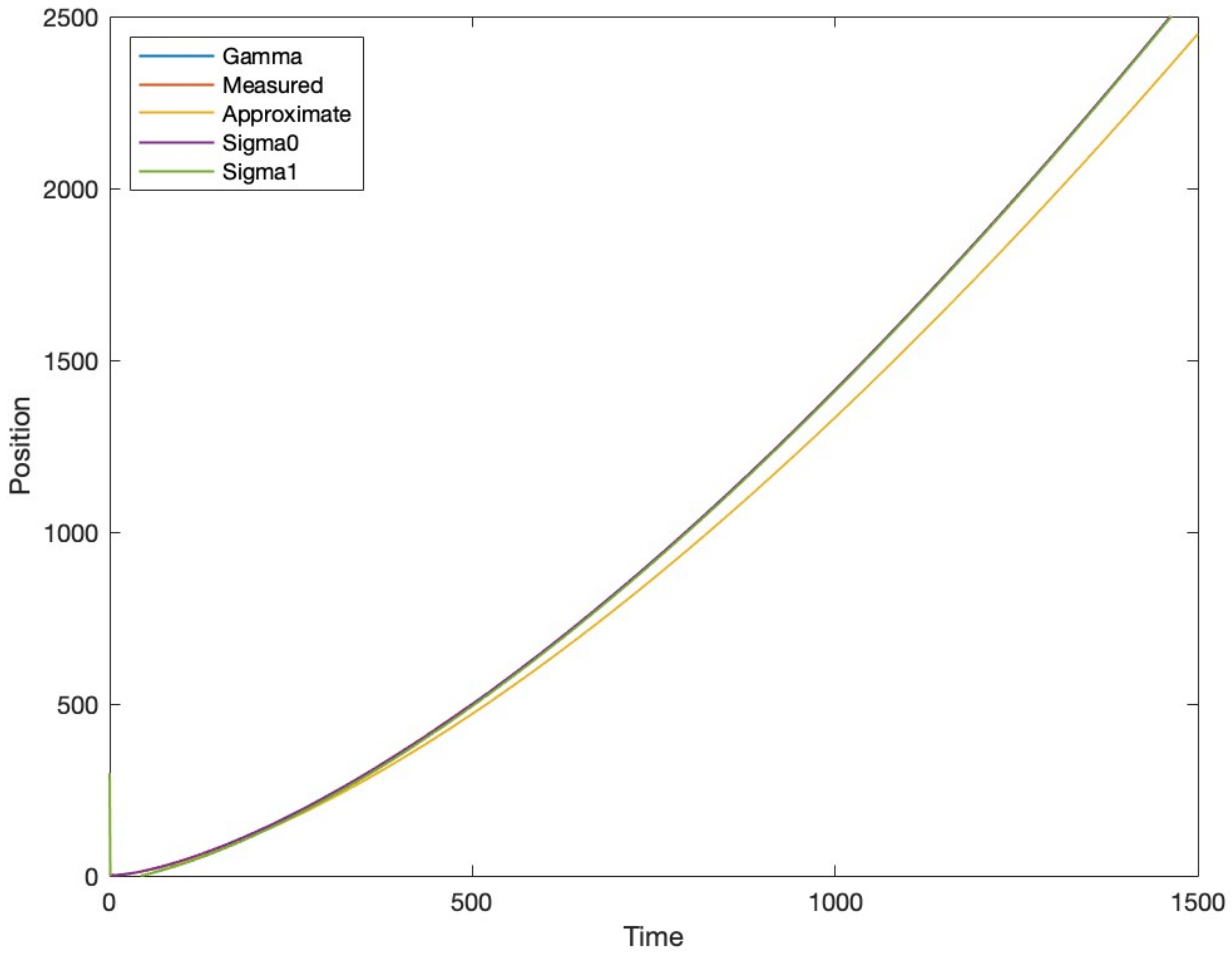
Homogeneous quenches: fronts

$$u_t = u_{xx} + \mu(t)u - u^3, \quad \mu(t) = \tanh(\epsilon t), \ \text{ or } \mu(t) = \epsilon t$$





Joint with B. Hosek, M. Avery, and 2 REU students: Odalys Garcia-Lopez, Ethan Shade, [in progress]

Naive Prediction: $x_f(t) \approx \int_0^t 2\sqrt{\mu(s)}ds$

Analysis of Green's Function of linearization $v_t = v_{xx} + \mu(t)v$: $x_f(t) \approx 2\left(t\int_0^t \mu(s)ds\right)^{1/2}$

Refined predictions using half-line analysis with Dirchlet BC and moving frame $y = x - x_f(t) > 0$:

$$\begin{cases} v_t &= v_{yy} + x_f'(t)v_y + \mu(t)v, \quad y > 0 \\ v &= 0, \quad y = 0 \end{cases}$$

See [Hamel,Nolen, Roquejoffre, Ryzhik, '13]

Homogeneous quenches: patterns

$$A_t = (1 + i\alpha)A_{xx} + \mu(t)A - (1 + i\beta)A\,|A\,|^2$$

Linearized Green's Function analysis, $v_t = (1 + i\alpha)v_{xx} + \mu(t)v$,

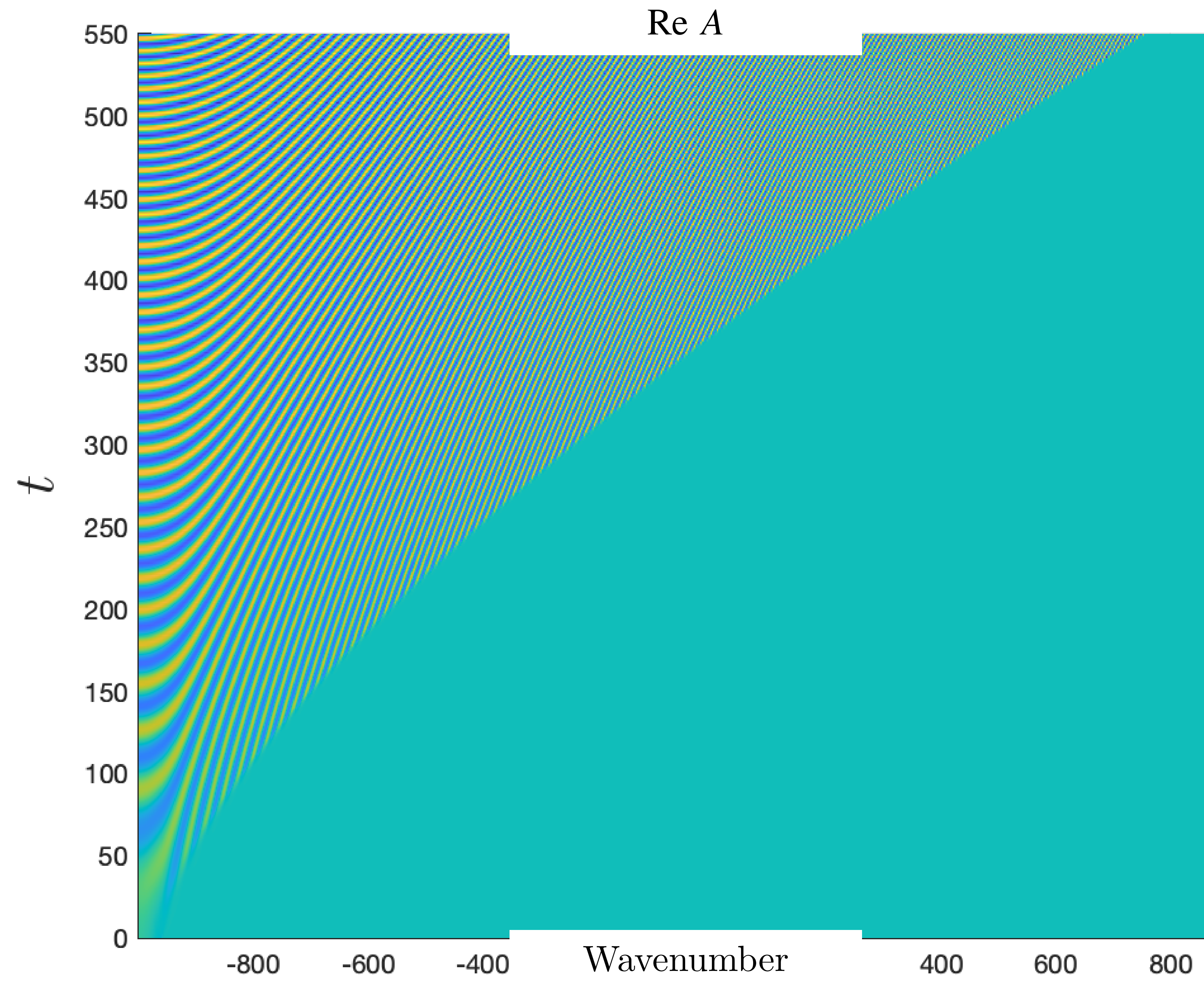
- $|\,v\,| = c \qquad x_f(t) \approx 2\left((1+\alpha^2)t\int_0^t \mu(s)ds\right)^{1/2}$

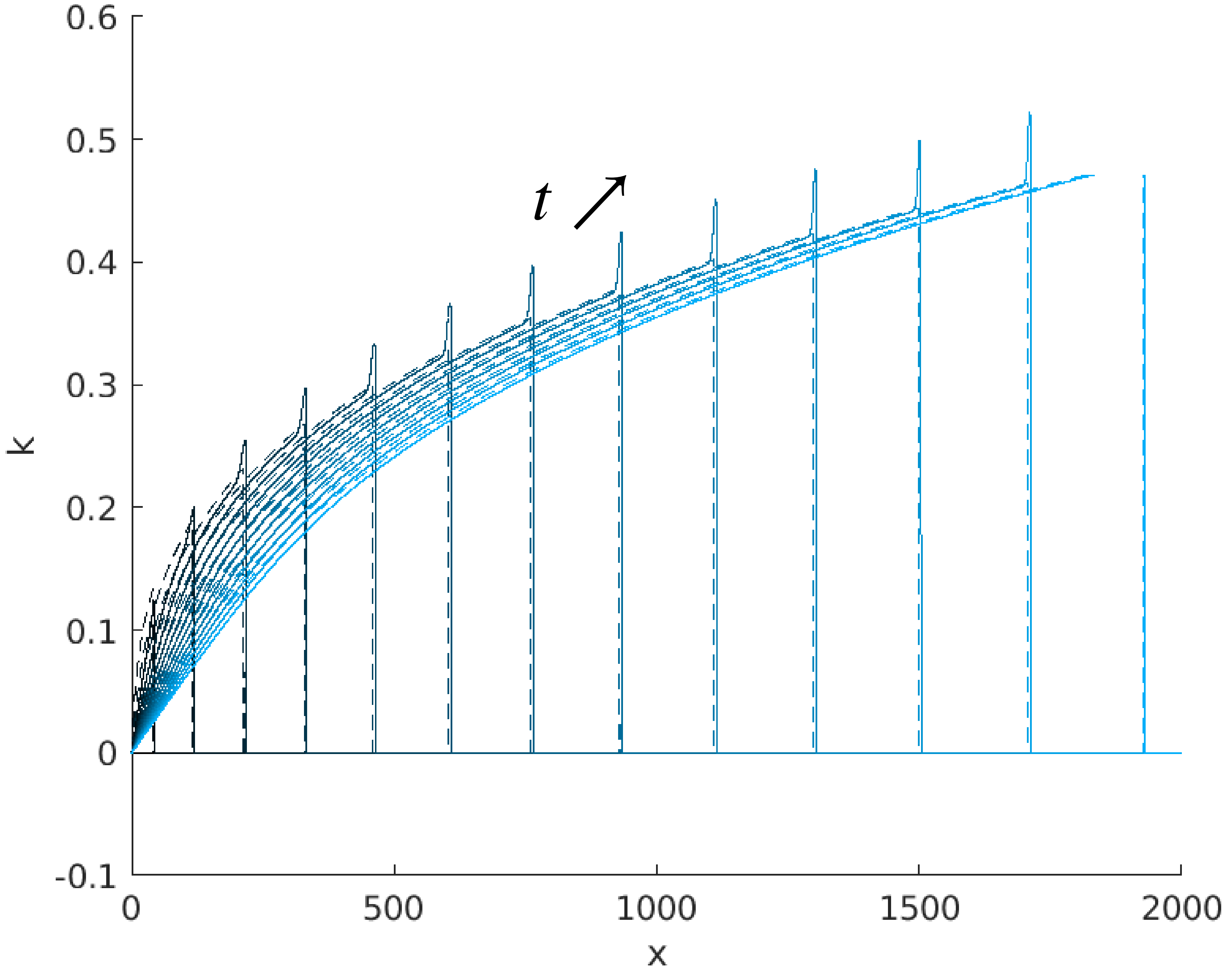
- Measure frequency $\arg v = \omega_f(t) = \alpha\mu(t)$

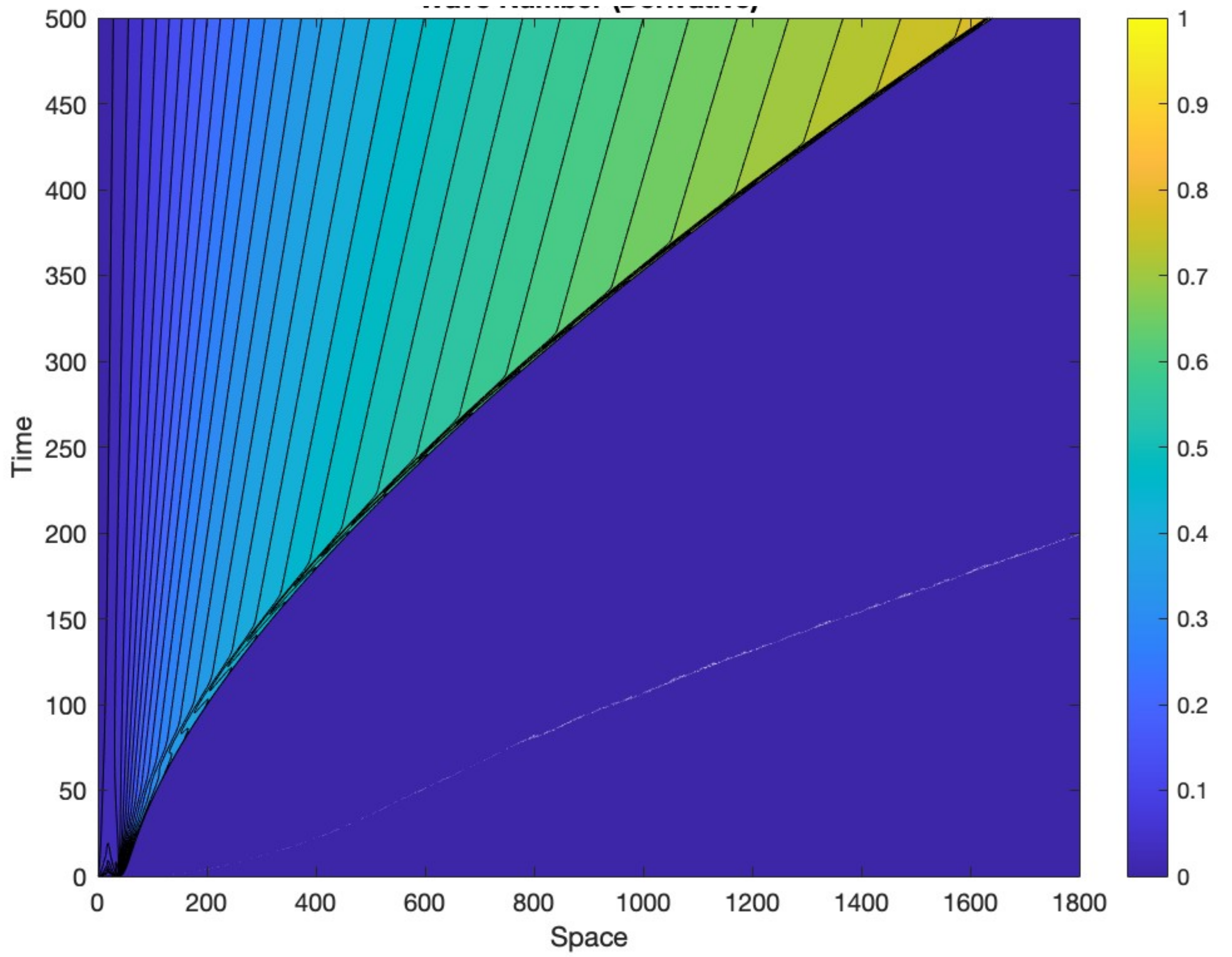
- Wavenumber prediction at leading edge through dispersions relation

$$\omega_f(t) = (\beta - \alpha)k_f(t)^2 + x_f'(t)k - \beta\mu(t)$$

- Burgers' modulation eqn. for bulk wavenumber dynamics







NSF

Thanks!

- RG, T.J. Kaper, A. Scheel, T. Vo, *Fronts in the wake of a parameter ramp: slow passage through pitchfork and fold bifurcations,* SIADS ‘23.
- RG, T.J. Kaper, A. Scheel, *Fronts in the wake of a parameter ramp: coherent structures in the critical scaling,* Stud. App. Math ‘24
- RG, A. Scheel, *Growing patterns,* Nonlinearity ‘23
- RG, B. Hosek, M. Avery, O. Garcia-Lopez, E. Shade *Invasion fronts under slow homogeneous quenching,* in progress

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