Periodic high contrast environments: Homogenization and beyond

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BU-Keio-Tsinghua Workshop 2024 on
Differential Equations, Dynamical Systems and Applied Mathematics

Boston University · Boston · May 28-31, 2024

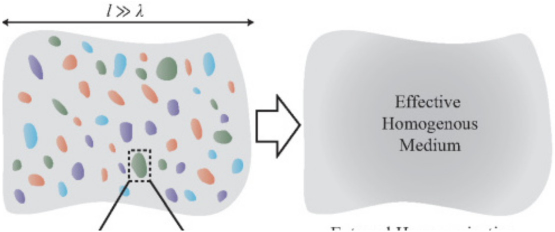
Outline

1. Backgrounds and set-up: Homogenization, high contrast limit
2. High contrast limits for elastostatics
3. Homogenization with high contrast parameter
4. Some proofs for the high contrast homogenization
5. A model of wave propagations in high contrast honeycomb structure

» Heterogeneous media are common in applications: small-scale variations in coefficients, in the geometry of the domains, and in other types of data.

» Difficulties to model, analyze, and compute...

» Homogenization, or effective medium theory: come up with a simplified model that qualifies as a *good* approximation.



A classical example

» Stationary conductivity in heterogeneous media:

$$-\nabla \cdot (A(\tfrac{x}{\varepsilon})\nabla u_\varepsilon) = f,$$

with proper boundary data.

» Structural assumptions: periodicity, stationary ergodicity, etc.

» Typical result: u_ε converges, as $\varepsilon \to 0$, to the solution $\overline{u}$ of

$$-\nabla \cdot (\overline{A}\nabla u) = f,$$

and the effective coefficient $\overline{A}$ is defined by a cell problem.

» Qualitative v.s. Quantitative results, and physical importances:

» Elliptic systems: $-\partial_i \left(a_{ij}^{\alpha\beta}(\tfrac{x}{\varepsilon})\partial_j u_\varepsilon^\beta \right) = f^\beta$, for u_ε valued in $\mathbb{R}^m$ with domain in $\mathbb{R}^d$.

A formal derivation, the cell problem

» Ansatz: $u_\varepsilon = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + ...$, $y = \frac{x}{\varepsilon}$, $y \in \mathbb{T}^d$.

» ∇ -> $\nabla_x + \frac{1}{\varepsilon}\nabla_y$, so

» $-\nabla \cdot (A(y)\nabla) \mapsto \varepsilon^{-2}\mathcal{L}_0 + \varepsilon^{-1}\mathcal{L}_1 + \mathcal{L}_2$, with

$$\begin{aligned}
\mathcal{L}_0 &= -\nabla_y \cdot (A(y)\nabla_y), \\
\mathcal{L}_1 &= -\nabla_y \cdot (A(y)\nabla_x) - \nabla_x \cdot (A(y)\nabla_y), \\
\mathcal{L}_2 &= -\nabla_x \cdot (A(y)\nabla_x).
\end{aligned}$$

» $\mathcal{L}_0(u_0) = -\nabla_y \cdot (A(y)\nabla_y u_0(x,y)) = 0$ in $\mathbb{T}^d$, yields $u_0(x,y) = u_0(x)$.

» $\mathcal{L}_0(u_1) = -\mathcal{L}_1(u_0) = (\nabla_y \cdot A(y))\nabla_x u_0$, yields $u_1(x,y) = \chi^j(y)\partial_{x_j} u_0$

» $\chi^j(y)$ solves cell problem

$$-\nabla_y \cdot \left(A(y)\nabla(\chi^j + e_j \cdot y)\right) = 0.$$

» Next order equation:

$$\begin{aligned}
\mathcal{L}_0(u_2) &= f - \mathcal{L}_1(u_1) - \mathcal{L}_2(u_0) \\
&= \nabla_x \cdot (A(y)\nabla_x u_0) + \nabla_y \cdot [A(y)\chi\nabla_x u_0] + \nabla_x \cdot [(A\nabla_y \chi)\nabla_x u] + f(x).
\end{aligned}$$

» Fredholm alternative yields solvability condition

$$\int_{\mathbb{T}^d} \nabla_x \cdot \{(A(y) + A(y)\nabla_y \chi(y))\, \nabla_x\} + f(x)\, dy = 0.$$

» Homogenized problem: $-\nabla_x \cdot \left(\overline{A}\nabla_x u_0(x)\right) = f$, with

$$\overline{A} = \int_{\mathbb{T}^d} A(y) + A(y)\nabla_y \chi(y)\, dy.$$

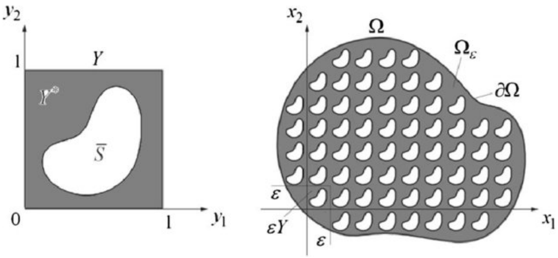
» Formally, the expansion yields $u_\varepsilon(x) \sim u_0(x) + \varepsilon\chi^j(\frac{x}{\varepsilon})\partial_{x_j} u_0(x) + h.o.t.$

Literature: Bensoussan-Lions-Papanicolaou, Tartar, Allaire, F.-H. Lin, Kenig, Shen, Geng, Zhuge, ... Papanicolaou-Varadhan, Jikov-Olenik-Kozlov, Gloria-Otto, Armstrong-Kuusi, ...

Periodic structure of inclusions

» Setting of this talk: periodic, high contrast coefficients; possible to generalize.

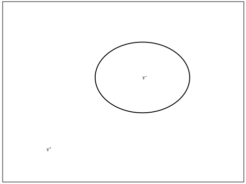
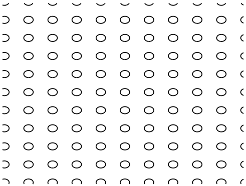
» Geometric configuration of high contrast inclusions (or, holes in 'perforated' domains):



Description of the periodic structure

» Unit cell: $Y = (-\frac{1}{2}, \frac{1}{2})^d$ the unit cube; $T \subset Y$ the unit hole, and the unit perforated cell $Y_f = Y \setminus \overline{T}$.

» Assume: ∂T is smooth, $\overline{B}_{1/8} \subset T, \overline{T} \subseteq B_{3/8}$...

» Array of inclusions/Perforated whole space at unit scale: $\mathbb{R}^d_f = \mathbb{R}^n \setminus (\cup_{k\in\mathbb{Z}^d} k + \overline{T})$, and the ε-scale version: $\varepsilon\mathbb{R}^d_f$.

» ε-scale perforated domain/or with inclusions: $\Omega_\varepsilon = \Omega \cap (\varepsilon\mathbb{R}^d_f)$...

» Type II domain: D_ε (holes inside) separated from $\partial\Omega$.

Model equation with high contrast inclusions

We consider the following transmission problem

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega\backslash\overline{D_\varepsilon}, \\ \mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{u} = 0 & \text{in } D_\varepsilon, \\ \mathbf{u}|_- = \mathbf{u}|_+ & \text{on } \partial D_\varepsilon, \\ \left.\dfrac{\partial\mathbf{u}}{\partial\nu_{(\widetilde{\lambda},\widetilde{\mu})}}\right|_- = \left.\dfrac{\partial\mathbf{u}}{\partial\nu_{(\lambda,\mu)}}\right|_+ & \text{on } \partial D_\varepsilon, \\ \left.\dfrac{\partial\mathbf{u}}{\partial\nu_{(\lambda,\mu)}}\right|_{\partial\Omega} = \mathbf{g} \in H_{\mathbf{R}}^{-\frac{1}{2}}(\partial\Omega), & \mathbf{u}|_{\partial\Omega} \in H_{\mathbf{R}}^{\frac{1}{2}}(\partial\Omega), \end{cases}$$

- The Lame pairs (λ,μ) and $(\widetilde{\lambda},\widetilde{\mu})$ satisfy $\mu_i > 0$, $3\lambda_i + 2\mu_i > 0$.
- $\frac{\partial\mathbf{u}}{\partial\nu_{(\lambda,\mu)}}\Big|_{\partial D_\varepsilon} := \lambda(\mathrm{div}\,\mathbf{u})\mathbf{N} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\mathbf{N}$ denotes the conormal derivative.

Homogenization with High Contrasts

» High contrast limit + homogenization, quantitative asymptotic analysis

$$\begin{array}{ccc} u^{\varepsilon,\mathrm{C}} & \xrightarrow{\text{hight contrast limit}} & u^{\varepsilon} \\ {\scriptstyle \varepsilon\to 0}\Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon\to 0} \\ u^{\mathrm{C}} & \longrightarrow & u \end{array}$$

Path I: High contrast limits

» Related parameters:

- E: Young's modulus;
- K: bulk modulus;
- $\widetilde{\mu}$: shear modulus.

$$E = \frac{\widetilde{\mu}(3\widetilde{\lambda} + 2\widetilde{\mu})}{\widetilde{\lambda} + \widetilde{\mu}} \quad \text{and} \quad K = \frac{3\widetilde{\lambda} + 2\widetilde{\mu}}{3},$$

» Some interesting asymptotic regimes:

- $\tilde{\lambda} \to \infty$, $\tilde{\mu}$ fixed: K of the inclusion tends to ∞, $E, \tilde{\mu}$ finte. (incompressible)
- $\tilde{\lambda}, \tilde{\mu} \to 0$, e.g., thin air, 'soft'
- $\tilde{\mu} \to \infty$, $\tilde{\lambda}$ fixed: 'hard' inclusions

Coupled Lamé-Stokes

The first extreme case yields a coupled Lamé-Stokes system:

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\mathbf{u} = 0 & \text{in } \Omega\backslash\overline{D_\varepsilon}, \\ \mathcal{L}_{\infty,\widetilde{\mu}}(\mathbf{u},p) = 0 & \text{in } D_\varepsilon, \\ \mathbf{u}|_- = \mathbf{u}|_+ & \text{on } \partial D_\varepsilon, \\ \left.\dfrac{\partial(\mathbf{u},p)}{\partial\nu_{(\infty,\widetilde{\mu})}}\right|_- = \left.\dfrac{\partial\mathbf{u}}{\partial\nu_{(\lambda,\mu)}}\right|_+ & \text{on } \partial D_\varepsilon, \\ \left.\dfrac{\partial\mathbf{u}}{\partial\nu_{(\lambda,\mu)}}\right|_{\partial\Omega} = \mathbf{g} \in H_{\mathbf{R}}^{-\frac{1}{2}}(\partial\Omega), & \mathbf{u}|_{\partial\Omega} \in H_{\mathbf{R}}^{\frac{1}{2}}(\partial\Omega), \end{cases}$$

» where $\mathcal{L}_{\infty,\widetilde{\mu}}$ denotes the Stokes system with viscosity constant $\widetilde{\mu}$, i.e.,

$$\mathcal{L}_{\infty,\widetilde{\mu}}(\mathbf{u},p) := \widetilde{\mu}\Delta\mathbf{u} + \nabla p,$$

with imcompressible condition $\operatorname{div}\mathbf{u} = 0$.

Soft and hard inclusions

» The second extreme setting, as $\widetilde{\lambda}$ and $\widetilde{\mu}$ both tend to zero, leads to

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\mathbf{u} = 0 & \text{in } \Omega\backslash\overline{D_\varepsilon}, \\ \left.\dfrac{\partial \mathbf{u}}{\partial \nu_{(\lambda,\mu)}}\right|_{+} = 0 & \text{on } \partial D_\varepsilon, \\ \left.\dfrac{\partial \mathbf{u}}{\partial \nu_{(\lambda,\mu)}}\right|_{\partial\Omega} = \mathbf{g} \in H_{\mathbf{R}}^{-\frac{1}{2}}(\partial\Omega), & \mathbf{u}|_{\partial\Omega} \in H_{\mathbf{R}}^{\frac{1}{2}}(\partial\Omega). \end{cases}$$

» The third one, as $\widetilde{\mu} \to \infty$ and $\widetilde{\lambda}$ fixed, leads to

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\mathbf{u} = 0 \quad \text{in } \Omega\backslash\overline{D_\varepsilon}, \\ \mathbf{u} \in \mathbf{R} \quad \text{in } D_\varepsilon, \\ \mathbf{u}|_{-} = \mathbf{u}|_{+} \quad \text{on } \partial D_\varepsilon, \\ \left.\dfrac{\partial \mathbf{u}}{\partial \nu_{(\lambda,\mu)}}\right|_{\partial\Omega} = \mathbf{g} \in H_{\mathbf{R}}^{-\frac{1}{2}}(\partial\Omega), \quad \mathbf{u}|_{\partial\Omega} \in H_{\mathbf{R}}^{\frac{1}{2}}(\partial\Omega), \end{cases}$$

Uniform convergence rates

Theorem (Fu-J. 2022)

In all three cases, let $\mathbf{u}_{\text{lim}}$ denote solution of the limiting problem. There exists $C > 0$ (which depends on inclusion shape and Ω) but independent of ε (periodicity) such that

$$\|\mathbf{u} - \mathbf{u}_{\text{lim}}\|_{H^1(\Omega \setminus \overline{D_\varepsilon})} \le \begin{cases} \dfrac{C}{\widetilde{\lambda}} \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\partial\Omega)} & \text{as } \widetilde{\lambda} \to \infty, \\ C(\widetilde{\lambda} + \widetilde{\mu}) \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\partial\Omega)} & \text{as } (\widetilde{\lambda}, \widetilde{\mu}) \to (0,0), \\ \dfrac{C}{\widetilde{\mu}} \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\partial\Omega)} & \text{as } \widetilde{\mu} \to \infty. \end{cases}$$

» Proofs based on layer potential tools

- All problems have layer potential representations;
- The usage of Neumann/Dirichlet (fundamental) functions, DtN operators;
- A uniform 'spectral gap' for the NP operator associated to ∂D_ε;

» Earlier works: Escauriaza and Seo, Dahlberg-Kenig-Verchota, Ammari et. al.

Path II: Homogenization with high contrast parameters

For simplicity consider only one contrast parameter δ, for the Dirichlet problem

$$\begin{cases} -\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = -\text{div}\left[\Lambda_\delta(\frac{x}{\varepsilon})A(\frac{x}{\varepsilon})\nabla u_{\varepsilon,\delta}\right] = f, & \text{in } \Omega, \\ u_{\varepsilon,\delta} = g, & \text{on } \partial\Omega. \end{cases}$$

» contrast parameter: δ, either $0 < \delta \ll 1$ or $\delta \gg 1$.

» high contrast inclusions modeled by: $\Lambda_\delta = \mathbf{1}_{Y\setminus\overline{T}}(y) + \delta\mathbf{1}_T(y)$ at the unit scale; extend, scale and cut a piece to get the ε-scale medium.

» standard assumptions on A: periodicity, uniform ellipticity (in background).

» system setting is fine, e.g., if $A = (a_{ij}^{\alpha\beta})$ has very strong ellipticity.

Homogenization with high contrast parameter δ

» Treat δ as a parameter, homogenized solution $\hat{u}_\delta$ given by

$$\begin{cases} -\mathcal{L}_{0,\delta}(\hat{u}_\delta) = -\mathrm{div}(\hat{A}_\delta \nabla \hat{u}_\delta) = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

» Homogenized coefficient: $\hat{A}_\delta = \int_Y \Lambda_\delta(y)A(y)\,[I + \nabla \chi_\delta]$, where

» $\chi_\delta = (\chi_\delta^1, \dots, \chi_\delta^d)$ solve cell problem:

$$-\mathcal{L}_{1,\delta}(\chi_\delta^i + y_i) = -\mathrm{div}(\Lambda_\delta(y)A(y)[\nabla \chi_\delta^i(y) + e_i]) = 0, \qquad \text{in } \mathbb{T}^d.$$

» Main questions:

- what are the convergence rates like?
- how does $\hat{A}_\delta$ (and, hence, $\hat{u}_\delta$) behave?
- do we have uniform regularity results, e.g., $u_{\varepsilon,\delta}$ is uniformly (in ε and also in δ) Lipschitz?

» For $\delta = 1$ (no high contrast), uniform regularity results date back to Avellaneda-Lin; recent developments by Zh. Shen.

Theorem (Fu-J. 2024+)

Let D_ε denote the collection of high contrast inclusions. For uniform elliptic A, we have

$$\|u_{\varepsilon,\delta} - \hat{u}_\delta - \varepsilon\chi_\delta(\tfrac{x}{\varepsilon})S_\varepsilon(\eta_\varepsilon\nabla\hat{u}_\delta) - \delta^{-1}\mathcal{L}_{D_\varepsilon}^{-1}(f)\|_{H^1(\Omega)} \le C\sqrt{\varepsilon}\left(\|f\|_{L^2(\Omega)} + \|\nabla_{\tan} g\|_{L^2(\partial\Omega)}\right).$$

Here $\mathcal{L}_{D_\varepsilon}^{-1}$ is the solution operator to $-\mathcal{L}_{\varepsilon,1}(v) = f$ in D_ε with zero Dirichlet boundary at ∂D_ε.

» For $\delta \ll 1$, $\delta^{-1}\mathcal{L}_{D_\varepsilon}^{-1}(f)$ is an additional corrector.

Uniform Lipschitz regularity

» Additional assumption: $A(y) \in C^{0,\alpha}(\mathbb{T}^d; \mathbb{M}_{\mathsf{sym}})$, i.e., A is Hölder in the unit scale. Also, assume T (model inclusion) has regular ($C^{1,\alpha}$) boundary.

» For $\delta \in (0,1)$, assume that $f = O(\delta)$ in D_ε.

» $f \in L^p$, $p > d$.

Theorem (Fu-J. 2024+)

Under the additional assumptions, if $\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = f$ in a boundary cylinder D_2, and $u_{\varepsilon,\delta} = g$ in Δ_2. Then there is a $c > 0$ for $c\varepsilon \le r < 1$,

$$\left(\fint_{D_r} |\nabla u_{\varepsilon,\delta}|^2\right)^{\frac{1}{2}} \le C\left\{\left(\fint_{D_2} |\nabla u_{\varepsilon,\delta}|^2\right)^{\frac{1}{2}} + \|f\|_{L^p(D_2)} + [\nabla_{\mathsf{tan}} g]_{C^{0,\rho}(\Delta_2)}\right\}.$$

» This is a large-scale (down to ε) Lipschitz estimate, but with the unit-scale regularity of A and by unit-scale Lipschitz estimate and rescaling, this is all one needs to get uniform Lipschitz.

» **Goal**: uniform in δ convergence rates $O(\sqrt{\varepsilon})$.

$$w_{\varepsilon,\delta} := u_{\varepsilon,\delta} - \widehat{u}_\delta - \varepsilon\chi^\alpha_{\delta,i}(x/\varepsilon)S_\varepsilon\left(\eta_\varepsilon\frac{\partial\widehat{u}^\alpha_\delta}{\partial x_i}\right),$$

Here, S_ε is a standard smoothing operator, η_ε is a cut-off ε-away from boundary.

$$\{\mathcal{L}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\}^\alpha = -\frac{\partial}{\partial x_i}\left(\Lambda_\delta(x/\varepsilon)P^\alpha_{\varepsilon,\delta,i}\right) - \frac{\partial}{\partial x_i}Q^\alpha_{\varepsilon,\delta,i},$$

where $P_{\varepsilon,\delta} = (P^\alpha_{\varepsilon,\delta,i})$ is defined by

$$P^\alpha_{\varepsilon,\delta,i} = a^{\alpha\beta}_{ij}\left(\frac{x}{\varepsilon}\right)\left[S_\varepsilon\left(\eta_\varepsilon\frac{\partial\widehat{u}^\beta_\delta}{\partial x_j}\right) - \frac{\partial\widehat{u}^\beta_\delta}{\partial x_j} - \varepsilon\chi^{\gamma\beta}_{\delta,k}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}S_\varepsilon\left(\eta_\varepsilon\frac{\partial\widehat{u}^\gamma_\delta}{\partial x_k}\right)\right]$$

and $Q_{\varepsilon,\delta} = (Q^\alpha_{\varepsilon,\delta,i})$ is defined by

$$Q^\alpha_{\varepsilon,\delta,i} = \varepsilon\Psi^{\alpha\beta}_{\delta,kij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_k}S_\varepsilon\left(\eta_\varepsilon\frac{\partial\widehat{u}^\beta_\delta}{\partial x_j}\right) - \widehat{a}^{\alpha\beta}_{\delta,ij}\left[S_\varepsilon\left(\eta_\varepsilon\frac{\partial\widehat{u}^\beta_\delta}{\partial x_j}\right) - \frac{\partial\widehat{u}^\beta_\delta}{\partial x_j}\right].$$

Basic estimates

$$\begin{cases} \{\Lambda_\delta(x/\varepsilon)\mathcal{L}_{\varepsilon,1}(w_{\varepsilon,\delta})\}^\alpha = -\Lambda_\delta(x/\varepsilon)\dfrac{\partial P^\alpha_{\varepsilon,\delta,i}}{\partial x_i} - \dfrac{\partial Q^\alpha_{\varepsilon,\delta,i}}{\partial x_i} & \text{in } \Omega \setminus \partial D_\varepsilon, \\ \left\{\delta \dfrac{\partial w_{\varepsilon,\delta}}{\partial \nu}\Big|_- - \dfrac{\partial w_{\varepsilon,\delta}}{\partial \nu}\Big|_+\right\}^\alpha = \delta P^\alpha_{\varepsilon,\delta,i} n_i\big|_- - P^\alpha_{\varepsilon,\delta,i} n_i\big|_+ & \text{on } \partial D_\varepsilon, \\ w_{\varepsilon,\delta} = 0 & \text{on } \partial\Omega. \end{cases}$$

» We have the following basic estimates:

$$\|\nabla w_{\varepsilon,\delta}\|_{L^2(\Omega_\varepsilon)} \le C\big(\|P_{\varepsilon,\delta}\|_{L^2(\Omega)} + \|Q_{\varepsilon,\delta}\|_{L^2(\Omega)}\big), \qquad \text{for } 0 < \delta < 1,$$

and

$$\|\nabla w_{\varepsilon,\delta}\|_{L^2(D_\varepsilon)} \le C\big(\|P_{\varepsilon,\delta}\|_{L^2(\Omega)} + \delta^{-\frac{1}{2}}\|Q_{\varepsilon,\delta}\|_{L^2(\Omega)}\big), \qquad \text{for } 1 < \delta < \infty.$$

Decomposition

» **Idea**: decomposition, use operator $\mathcal{L}_{\varepsilon,1}$.

Let $w^1_{\varepsilon,\delta} \in H^1_0(\Omega;\mathbb{R}^m)$ be the solution of

$$\begin{cases} \{\mathcal{L}_{\varepsilon,1}(w^1_{\varepsilon,\delta})\}^\alpha = -\dfrac{\partial P^\alpha_{\varepsilon,\delta,i}}{\partial x_i} - \dfrac{\partial Q^\alpha_{\varepsilon,\delta,i}}{\partial x_i} & \text{in } \Omega, \\ w^1_{\varepsilon,\delta} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $w^2_{\varepsilon,\delta} \in H^1(D_\varepsilon;\mathbb{R}^m)$ be the solution of

$$\begin{cases} \{\mathcal{L}_{\varepsilon,1}(w^2_{\varepsilon,\delta})\}^\alpha = -\dfrac{\partial P^\alpha_{\varepsilon,\delta,i}}{\partial x_i} - \dfrac{1}{\delta}\dfrac{\partial Q^\alpha_{\varepsilon,\delta,i}}{\partial x_i} & \text{in } D_\varepsilon, \\ w^2_{\varepsilon,\delta} = w^1_{\varepsilon,\delta} & \text{on } \partial D_\varepsilon. \end{cases}$$

» By computation,

$$-\frac{\partial Q_{\varepsilon,\delta,i}}{\partial x_i} = \delta R^{\alpha}_{\varepsilon,\delta} + f \qquad \text{in } D_\varepsilon,$$

with

$$R^{\alpha}_{\varepsilon,\delta} = \left[a^{\alpha\beta}_{ij}\left(\frac{x}{\varepsilon}\right) + a^{\alpha\gamma}_{ik}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_k}\chi^{\gamma\beta}_{\delta,j}\left(\frac{x}{\varepsilon}\right)\right]\frac{\partial}{\partial x_i} S_\varepsilon\left(\eta_\varepsilon \frac{\partial \widehat{u}^{\beta}_{\delta}}{\partial x_j}\right).$$

» Note that

$$\begin{cases} \mathcal{L}_{\varepsilon,1}\left(w^2_{\varepsilon,\delta} - \delta^{-1}\mathcal{L}^{-1}_{D_\varepsilon}(F) - w^1_{\varepsilon,\delta}\right) = -\frac{\partial Q_{\varepsilon,\delta,i}}{\partial x_i} + R_{\varepsilon,\delta} & \text{in } D_\varepsilon, \\ w^2_{\varepsilon,\delta} - \delta^{-1}\mathcal{L}^{-1}_{D_\varepsilon}(f) - w^1_{\varepsilon,\delta} = 0 & \text{on } \partial D_\varepsilon. \end{cases}$$

» Good estimates:

$$\|\nabla w^1_{\varepsilon,\delta}\|_{L^2(\Omega)} \le C\big(\|P_{\varepsilon,\delta}\|_{L^2(\Omega)} + \|Q_{\varepsilon,\delta}\|_{L^2(\Omega)}\big),$$

$$\big\|\nabla\big(w^2_{\varepsilon,\delta} - \delta^{-1}\mathcal{L}^{-1}_{D_\varepsilon}(f)\big)\big\|_{L^2(D_\varepsilon)} \le C\big(\|P_{\varepsilon,\delta}\|_{L^2(\Omega)} + \|Q_{\varepsilon,\delta}\|_{L^2(\Omega)} + \varepsilon\|R_{\varepsilon,\delta}\|_{L^2(\Omega)}\big).$$

An energy-balance lemma

» Decompose $w_{\varepsilon,\delta}$ into $w^{\mathsf{R}}_{\varepsilon,\delta} = w^1_{\varepsilon,\delta}\mathbf{1}_{\Omega_\varepsilon} + w^2_{\varepsilon,\delta}\mathbf{1}_{D_\varepsilon}$ and $w^{\mathsf{S}}_{\varepsilon,\delta} := w_{\varepsilon,\delta} - w^{\mathsf{R}}_{\varepsilon,\delta}$. Then:

By definition, $w^{\mathsf{S}}_{\varepsilon,\delta} \in H^1_0(\Omega;\mathbb{R}^m)$ and solves

$$\mathcal{L}_{\varepsilon,1}(w^{\mathsf{S}}_{\varepsilon,\delta}) = 0 \qquad \text{in } \Omega \setminus \partial D_\varepsilon.$$

» A key estimate is:

Lemma

Under the periodicity and ellipticity,

$$C\int_{D_\varepsilon} |\nabla w^{\mathsf{S}}_{\varepsilon,\delta}|^2\,dx \le \int_{\Omega_\varepsilon} |\nabla w^{\mathsf{S}}_{\varepsilon,\delta}|^2\,dx \le C^{-1}\int_{D_\varepsilon} |\nabla w^{\mathsf{S}}_{\varepsilon,\delta}|^2\,dx.$$

» The key observation is: along each ∂T_ε (boundary of obstacle), we have

$$\int_{\partial T_\varepsilon} \frac{\partial w^{\mathsf{S}}_{\varepsilon,\delta}}{\partial \nu}\Big|_+ = 0.$$

So we may extend $w^{\mathsf{S}}_{\varepsilon,\delta}$ from inside to outside, and use energy estimate (variation principle) to get the second inequality. The first inequality involves a more direct extension from outside to inside.

Proof of Convergence rate

» By the decomposition, $w_{\varepsilon,\delta} = w^{\mathsf{R}}_{\varepsilon,\delta} + w^{\mathsf{S}}_{\varepsilon,\delta}$.

» *Case 1:* $0 < \delta < 1$. By the energy balance lemma, we have

$$\begin{aligned}\|\nabla w^{\mathsf{S}}_{\varepsilon,\delta}\|_{L^2(\Omega)} \le C\|\nabla w^{\mathsf{S}}_{\varepsilon,\delta}\|_{L^2(\Omega_\varepsilon)} &\le C\|\nabla w^{\mathsf{R}}_{\varepsilon,\delta}\|_{L^2(\Omega_\varepsilon)} + C\|\nabla w_{\varepsilon,\delta}\|_{L^2(\Omega_\varepsilon)} \\ &\le C\big(\|P_{\varepsilon,\delta}\|_{L^2(\Omega)} + \|Q_{\varepsilon,\delta}\|_{L^2(\Omega)} + \varepsilon\|R_{\varepsilon,\delta}\|_{L^2(\Omega)}\big),\end{aligned}$$

» *Case 2:* $1 \le \delta < \infty$. By the energy balance lemma, we have

$$\begin{aligned}\|\nabla w^{\mathsf{S}}_{\varepsilon,\delta}\|_{L^2(\Omega)} \le C\|\nabla w^{\mathsf{S}}_{\varepsilon,\delta}\|_{L^2(D_\varepsilon)} &\le C\|\nabla\big(w^{\mathsf{R}}_{\varepsilon,\delta} - \delta^{-1}\mathcal{L}^{-1}_{D_\varepsilon}(f)\big)\|_{L^2(D_\varepsilon)} \\ &\quad + C\delta^{-1}\|\nabla\mathcal{L}^{-1}_{D_\varepsilon}(f)\|_{L^2(D_\varepsilon)} + C\|\nabla w_{\varepsilon,\delta}\|_{L^2(D_\varepsilon)} \\ &\le C\big(\|P_{\varepsilon,\delta}\|_{L^2(\Omega)} + \|Q_{\varepsilon,\delta}\|_{L^2(\Omega)} + \varepsilon\|R_{\varepsilon,\delta}\|_{L^2(\Omega)} + \varepsilon\|f\|_{L^2(\Omega)}\big),\end{aligned}$$

» Conclusion:

$$\begin{aligned}&\|\nabla\big(w_{\varepsilon,\delta} - \delta^{-1}\mathcal{L}^{-1}_{D_\varepsilon}(F)\big)\|_{L^2(\Omega)} \\ \le& C\big(\|P_{\varepsilon,\delta}\|_{L^2(\Omega)} + \|Q_{\varepsilon,\delta}\|_{L^2(\Omega)} + \varepsilon\|R_{\varepsilon,\delta}\|_{L^2(\Omega)} + \varepsilon\|F\|_{L^2(\Omega)}\big).\end{aligned}$$

Key steps in the proof of Uniform Lipschitz regularity

· The Lipschitz regularity can be proved using Campanato-type scheme based on first order approximation, due to Armstrong-Shen, and an alternative to the method of Avellaneda-Lin.

Theorem (Shen 2021, Fu-J. 2024+(Boundary Caccioppoli inequalities))

(i) *(small δ) Let $R \geq 4\varepsilon$ and $\mathbf{D}_R$ be a boundary cylinder. If $\mathcal{L}_{\varepsilon,\delta}(u) = f_{\varepsilon,\delta}$ in $\mathbf{D}_{2R}$ and $u = g$ in Δ_{2R}, then*

$$\fint_{\mathbf{D}_{3R/2}} |\nabla u|^2 \leq C\Big\{ \frac{1}{R^2} \fint_{\mathbf{D}_{2R}} |u|^2 + R^2 \fint_{\mathbf{D}_{2R}} |f|^2 + \frac{1}{R^2} \|g\|^2_{L^\infty(\Delta_{2R})} + \|\nabla_{\tan} g\|^2_{L^\infty(\Delta_{2R})} \Big\}.$$

(ii) *(large δ) for positive integer ℓ,*

$$\int_{\mathbf{D}_{3R/2}} |\nabla u|^2 \leq \frac{C_\ell}{R^2} \int_{\mathbf{D}_{2R}} |u|^2 + C_\ell \left(\frac{\varepsilon}{R}\right)^{2\ell} \int_{\mathbf{D}_{2R}} |\nabla u|^2 + C_\ell \left(R^2 \int_{\mathbf{D}_{2R}} |f|^2 + \frac{1}{R^2} \|g\|^2_{L^\infty(\Delta_{2R})} + \|\nabla_{\tan} g\|^2_{L^\infty(\Delta_{2R})} \right).$$

First order approximation due to quantitative homogenization

Lemma

Suppose that $u_{\varepsilon,\delta}$ is a weak solution of $\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = f_{\varepsilon,\delta}$ in $\mathbf{D}_{2r}$ with $u_{\varepsilon,\delta} = g$ on Δ_{2r} for some $r \geq 4\varepsilon$. There exists $v \in H^1(\mathbf{D}_r)$ such that $\widehat{\mathcal{L}}_\delta(v) = f_{\varepsilon,\delta}$ in $\mathbf{D}_r$ with $v = g$ on Δ_r, and

$$\|u_{\varepsilon,\delta} - v\|_{\underline{L}^2(\mathbf{D}_r)} \leq C\left(\frac{\varepsilon}{r}\right)^{1/2}\Big\{\|u_{\varepsilon,\delta}\|_{\underline{L}^2(\mathbf{D}_{2r})} + r^2\|f\|_{\underline{L}^2(\mathbf{D}_{2r})} + \|g\|_{L^\infty(\Delta_{2r})} + \\ + r\|\nabla_{\tan} g\|_{L^\infty(\mathbf{D}_{2r})}\Big\} + C\left(\frac{\varepsilon}{r}\right)^{3/2} r\|\nabla u_{\varepsilon,\delta}\|_{\underline{L}^2(\mathbf{D}_{2r})}.$$

Consider the function $H(t)$ (and its minimizer $h(t) = |E_t|$ for the minimizer) that measures the approximation at scale t:

$$H[w](t) = \frac{1}{t}\inf_{\substack{E\in\mathbb{R}^{m\times d}\\ q\in\mathbb{R}^m}}\Bigg\{\left(⨍_{\mathbf{D}_t}|w - Ex - q|^2\right)^{\frac{1}{2}} + t^2\left(⨍_{\mathbf{D}_t}|f|^p\right)^{\frac{1}{p}} + \|g - Ex - q\|_{L^\infty(\Delta_t)} \\ + t\|\nabla_{\tan}(g - Ex)\|_{L^\infty(\Delta_t)} + t^{1+\rho}|\nabla_{\tan}(g - Ex)|_{C^{0,\rho}(\Delta_t)}\Bigg\},$$

Lemma

There is a $\theta \in (0,1)$ *and* $c > 0$, $c\theta \geq 4$*, and: Suppose* $u_{\varepsilon,\delta}$ *is a weak solution of* $\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = f_{\varepsilon,\delta}$ *in* $\mathbf{D}_{2r}$ *with* $u_{\varepsilon,\delta} = g$ *on* Δ_{2r} *for some* $r \geq c\varepsilon$*, then*

$$H[u_{\varepsilon,\delta}](\theta r) \leq \frac{1}{2} H[u_{\varepsilon,\delta}](r) + C\left(\frac{\varepsilon}{r}\right)^{1/2}\Big(H[u_{\varepsilon,\delta}](2r) + h[u_{\varepsilon,\delta}](2r)\Big) + C\left(\frac{\varepsilon}{r}\right)^{3/2}\|\nabla u_{\varepsilon,\delta}\|_{\underline{L}^2(\mathbf{D}_{2r})}.$$

· For homogenized equation, $H[v](\theta r) \leq \frac{1}{2} H[v](r)$.
· The above are the key because, by iterating the Caccioppoli inequality, we have the control for all $r \geq c\varepsilon$,

$$\|\nabla u\|_{\underline{L}^2(\mathbf{D}_r)} \leq C\left(\|\nabla u\|_{\underline{L}^2(\mathbf{D}_2)} + \|f\|_{\underline{L}^p(\mathbf{D}_2)} + \|\nabla_{\tan} g\|_{C^{0,\rho}(\Delta_2)}\right) + C\int_r^2 \frac{H(t)}{t}\,dt.$$

Consider the wave propagation problem:

$$\begin{cases} \left(\partial_t^2 + \mathcal{L}_{\varepsilon,\delta}\right) w_{\varepsilon,\delta} = 0 \quad \text{in } \mathbb{R}^2 \times (0,\infty), \\ w_{\varepsilon,\delta}(x,0) = F_1(x) S_1\left(\frac{x}{\varepsilon}\right) + F_2(x) S_2\left(\frac{x}{\varepsilon}\right), \\ \partial_t w_{\varepsilon,\delta}(x,0) = \mathrm{i}\frac{\omega_\delta^*}{\varepsilon}\left(F_1(x) S_1\left(\frac{x}{\varepsilon}\right) + F_2(x) S_2\left(\frac{x}{\varepsilon}\right)\right). \end{cases}$$

· $\mathcal{L}_{\varepsilon,\delta} = -\sigma_\delta(\cdot/\varepsilon)\nabla \cdot \left(\sigma_\delta^{-1}(\cdot/\varepsilon)\nabla\right)$.

· Triangular lattice: $\Lambda = \mathbb{Z}l_1 \oplus \mathbb{Z}l_2$, $l_1 = L(\sqrt{3}/2, 1/2)$, $l_2 = L(\sqrt{3}/2, -1/2)$.

· Dual lattice: $\Lambda^* = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$, $\alpha_i \cdot l_j = 2\pi\delta_{ij}$. $|Y^*| = 1$.

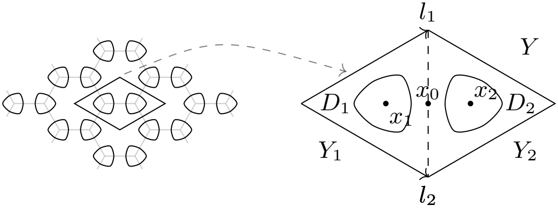


Figure: Honeycomb crystal.

Band Structure / Dispersion surfaces

Due to the periodic structure,

- $\mathcal{L}_\delta$ acting on $L^2_\delta(\mathbb{R}^2)$ (σ_δ^{-1}-weighted L^2) has a direct integral representation over $\alpha \in Y^*$.
- For each $\alpha \in Y^*$ (or over Brillouin zone), $\mathcal{L}_\delta$ acting on $L^2_\delta(\alpha)$ (subject to α-quasi-periodic condition $f(x+l) = e^{i\alpha \cdot l} f(x)$) is self-adjoint.
- For each $\alpha \in Y^*$, eigen-pairs $\{(\omega_j^2(\alpha), \Phi_j(\cdot, \alpha))\}_j$.
- Band: $\alpha \mapsto \omega_j(\alpha)$.

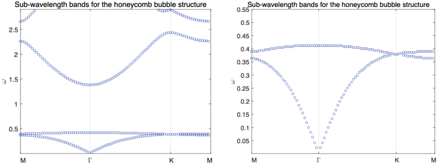
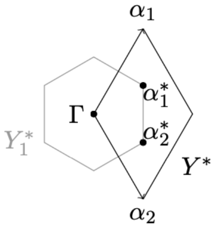


Figure 2: (left) The band structure of a bubbly honeycomb phononic crystal with $R = 1/50$ and $\delta = 1/9000$. The distance between the adjacent bubbles is one. (right) The band structure upon zooming in on the subwavelength region.



Dirac cone / Resonant modes

- Fefferman-Weinstein (2012) established (based on PDE models) theory of Dirac points at the high symmetry point α^* (in plot, K).
- Ammari et. al. (2020), Cassier-Weinstein (2022), Dirac points for large contrast.
- The high contrast case: sub-wavelength inclusions; applications in applied physics.
- $\omega_\pm^2(\alpha) = \omega^2(\alpha^*) \pm \delta \mathsf{c}|\alpha - \alpha^*|(1 + \mathcal{O}(|\alpha - \alpha^*|))$.
- $\omega_\delta^* = \sqrt{\frac{c_1^{\alpha^*}}{|D_1|}}\delta^{\frac{1}{2}} + \mathcal{O}(\delta), \qquad \lambda_\delta = \lambda_0 |c| \delta^{\frac{1}{2}} + \mathcal{O}(\delta), \qquad \lambda_0 = \frac{1}{2}\sqrt{\frac{1}{|D_1| c_1^{\alpha^*}}}$.

Theorem

At the Dirac point α^*, for small δ, one finds a pair of independent Bloch eigenmodes $S_{1,\delta}$ and $S_{2,\delta}$ satisfying:

(a) $\mathcal{R} S_{1,\delta} = \tau S_{1,\delta}, \qquad \mathcal{R} S_{2,\delta} = \overline{\tau} S_{2,\delta}, \qquad \mathcal{P}\mathcal{C} S_{1,\delta} = S_{2,\delta}$.

(b) $\|S_{1,\delta}\|_{L^2(Y)} = \|S_{2,\delta}\|_{L^2(Y)} = 1, \langle S_{1,\delta}, S_{2,\delta}\rangle_{L^2(Y \setminus \overline{D})} = \langle S_{1,\delta}, S_{2,\delta}\rangle_{L^2(D_1)} = \langle S_{1,\delta}, S_{2,\delta}\rangle_{L^2(D_2)} = 0$.

(c) *$S_{1,\delta}$ and $S_{2,\delta}$ are well approximated by S_1 and S_2 (via layer potential and capacitance matrices): $S_{j,\delta} - S_j = \mathcal{O}_{H^1(Y)}(\delta), \qquad$ for $j = 1, 2$.*

Approximation of the wave propagation

The wave propagation corresponds to the evolution of wave packets spectrally concentrated at the Dirac modes at the Dirac point.
Goal: approximation of the wave

$$w_{\varepsilon,\delta}(x,t) = e^{\mathrm{i}\frac{\omega_\delta^*}{\varepsilon}t}\left(V_1(x,t)S_1\left(\frac{x}{\varepsilon}\right) + V_2(x,t)S_2\left(\frac{x}{\varepsilon}\right)\right) + r_{\varepsilon,\delta}(x,t).$$

which remain wave packets spectrally concentrated with profiles evolving via

$$2\mathrm{i}\omega_\delta^*\partial_t\begin{pmatrix}V_1\\V_2\end{pmatrix} = \begin{pmatrix} & a_\delta(\partial_1+\mathrm{i}\partial_2)\\ -\overline{a_\delta}(\partial_1-\mathrm{i}\partial_2) & \end{pmatrix}\begin{pmatrix}V_1\\V_2\end{pmatrix},$$
$$(V_1,V_2)(x,0) = (F_1,F_2),$$

with estimates: for some positive ρ, ν, and C independent of ε,

$$\sup_{0\le t\le\varepsilon^{-\rho}} \|r_{\varepsilon,\delta}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C\varepsilon^\nu$$

Thank you very much for your attention !

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Path III: Homogenization of high contrast limit models

» As a sample homogenization problem for the equation after high contrast limit, we consider the homogenization of linear elastostatics in perforated domains:

$$\begin{cases} -\mathcal{L}^{\lambda,\mu}[u_\varepsilon](x) = -(\mu\Delta + (\lambda+\mu)\nabla\text{div})[u_\varepsilon] = f(x) \text{ in } \Omega^\varepsilon, \\ u_\varepsilon(x) = 0 \text{ in } \partial\Omega^\varepsilon. \end{cases}$$

» Similar problems: conductivity, or Stokes flow

$$\begin{cases} -\Delta u_\varepsilon = f \text{ in } \Omega^\varepsilon, \\ u_\varepsilon = 0 \text{ in } \partial\Omega^\varepsilon. \end{cases} \quad \text{or} \quad \begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f(x) \text{ in } \Omega^\varepsilon \\ \nabla\cdot u_\varepsilon = 0 \text{ in } \Omega^\varepsilon \\ u_\varepsilon(x) = 0 \text{ in } \partial\Omega^\varepsilon. \end{cases}$$

» At boundary of the holes, Dirichlet data are imposed !

» The domain Ω^ε is really $\Omega^{\varepsilon,\eta}$ and η may depends on ε.

Two-scale expansion: formal arguments

» A powerful *yet* formal approach: two-scale expansion with ansatz

$$u_\varepsilon = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \cdots,$$

Here, $u_i : \mathbb{R}^d \times Y_f \to \mathbb{R}$, $i = 0, 1, 2, \cdots$, the mapping $y \mapsto u_i(x, y)$ is periodic, u_i vanishes for $y \in \partial T$.

» Replace ∇ by $\nabla_x + \frac{1}{\varepsilon}\nabla_y$; differential operator replaced by

$$\frac{1}{\varepsilon^2}\mathcal{L}_y^{\lambda,\mu} + \frac{1}{\varepsilon}\left(2\mu\nabla_x \cdot \nabla_y + (\lambda+\mu)[\nabla_x \mathrm{div}_y + \nabla_y \mathrm{div}_x]\right) + \mathcal{L}_x^{\lambda,\mu}$$

» Plug in the ansatz, note $u_i(x, y) = 0$ for $y \in T$. We get:

$$\begin{aligned}
-\mathcal{L}_y^{\lambda,\mu}(u_0(x,y)) &= 0, \quad (\Rightarrow u_0 = 0)\\
-\mathcal{L}_y^{\lambda,\mu}(u_1(x,y)) &= 0, \quad (\Rightarrow u_1 = 0)\\
-\mathcal{L}_y^{\lambda,\mu}(u_2(x,y)) &= f(x), \quad (\Rightarrow u_2(x,y) = \textstyle\sum_k f_k(x)\chi_k(y)).
\end{aligned}$$

» χ_k, for each $k \in \{1, \cdots, d\}$, solves the cell problem:

$$\begin{cases} -\mathcal{L}_y^{\lambda,\mu} \chi_k(y) = e_k, & y \in Y_f, \\ \chi_k(y) = 0, & y \in \partial T. \end{cases}$$

» Formally, $\frac{u_\varepsilon}{\varepsilon^2} \approx \chi(\frac{x}{\varepsilon}) f(x)$, and hence weakly converges to Af, with $A := \fint_Y \chi$.

in the setting of $\eta = 1$, the homogenized equation is given by an algebraic formula.

» Darcy's law: $\frac{u_\varepsilon}{\varepsilon^2} \approx u_2(x, \frac{x}{\varepsilon}) = A(\frac{x}{\varepsilon})(f - \nabla p_0)$, where A is a periodic matrix with columns χ_k's. Riemann-Lebesgue yields

$$\frac{u_\varepsilon}{\varepsilon^2} \to \overline{A}(f - \nabla p_0), \qquad \nabla \cdot (\overline{A}(f - \nabla p_0)) = 0.$$

» (Formal expansion to Darcy's law by Keller, proof by Tartar.)

Dilute holes

» The unit perforated cell $Y_f = Y \setminus (\eta \overline{T})$. At ε-scale: hole size is $\eta\varepsilon$;.

» Dilute: $\eta = \eta_\varepsilon$ vanishes as $\varepsilon \to 0$.

» A Poincaré inequality: if $u \in H^1(B_R)$ and $u = 0$ in B_r, with $0 < r < R$. Then there exists $C = C(d)$ s.t.

$$\|u\|_{L^2(B_R)} \le \begin{cases} CR(\frac{R}{r})^{\frac{d-2}{2}} \|\nabla u\|_{L^2(B_R)}, & d \ge 3, \\ CR(\log \frac{R}{r})^{\frac{1}{2}} \|\nabla u\|_{L^2(B_R)}, & d = 2. \end{cases}$$

» Above applied in ε-cell: the bounding constant becomes

$$\sigma_\varepsilon = \begin{cases} \varepsilon \eta^{-\frac{d-2}{2}}, & d \ge 3, \\ \varepsilon |\log \eta|^{\frac{1}{2}}, & d = 2. \end{cases}$$

» Three asymptotic regimes: as $\varepsilon \to 0$,

Critical setting: $\sigma_\varepsilon \to \sigma_0$, $\sigma_0 > 0$. (holes w. critical smallness)

Super-critical setting: $\sigma_\varepsilon \to 0$. (larger holes)

Sub-critical setting: $\sigma_\varepsilon \to \infty$. (tiny holes)

Strange terms from nowhere

» Cioranescu and Murat first showed, for the conductivity problem and in the critical setting, trivial extension of u_ε converges weakly in $H_0^1(\Omega)$ to the solution of

$$-\Delta u + \text{Cap}(T)u = f \qquad \text{in } \Omega.$$

» Un terme étrange venu d'ailleurs, Cap(T): Newtonian capacity of T (for $d = 3$); term replaced by $\frac{1}{2\pi}u$ (for $d = 2$).

» Allaire then established the theory in the setting for Stokes flow. Effective model in the critical setting: Brinkman's law

$$-\Delta u + \nabla p + Mu = f, \qquad \nabla \cdot u = 0 \quad \text{in } D, \quad u = 0 \text{ on } \partial D.$$

» Effective model for Sub-critical: unperturbed; for Super-critical: Darcy's law (with permeability matrix M^{-1}).

» The matrix M is associated to Stokes system posed in $\mathbb{R}^d \setminus \overline{T}$ (for $d \geq 3$), and $M = \frac{1}{4\pi}I$ (for $d = 2$).

The version for Lamé system

» We find an M, positive definite, associated to "elastic capacity" / exterior problems (for $d \geq 3$) and $M = \frac{c_1(\lambda,\mu)}{2\pi} I$ (for $d = 2$)

Theorem (J. '21)

Qualitative homogenization results for three dilute regimes.
(i) If $\sigma_\varepsilon \to \infty$ (subcritical), then $\tilde{u}_\varepsilon$ converges weakly in $H_0^1(\Omega)$ to the solution of

$$-\mathscr{L}^{\lambda,\mu}[u] = f \quad \text{in } \Omega.$$

(ii) If $\sigma_\varepsilon \to \sigma_0 \in (0,\infty)$ (critical), then $\tilde{u}_\varepsilon$ converges weakly in $H_0^1(\Omega)$ to

$$-\mathscr{L}^{\lambda,\mu}[u] + \frac{M}{\sigma_0^2} u = f \quad \text{in } \Omega.$$

(iii) If $\sigma_\varepsilon \to 0$ (super-critical), $\frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2}$ converges weakly in L^2 to $M^{-1} f$.

» Cioranescu-Murat and Allaire proved their results through an abstract framework, and technical constructions built to apply the framework.

» Cell problems do not enter, at least not explicitly.

» In J. (2020), we developed a new approach based on cell problems (w. η-dependence), for the Poisson problem, w. advantages:

- natural, follows from two-scale expansion
- quantifiable
- yields correctors
- unified for all dilute regimes, for all $d \geq 2$.

» The approach works also for Lamé (this talk) and for Stokes systems.

The rescaled cell problems

» v_k^ε's are rescaled cell problem solutions.

» In general, $\eta = \eta_\varepsilon$, so the cell problem for χ_k is posed on $\mathbb{T}^d \setminus \eta\overline{T}$. Consider $\chi_k^\eta = \eta^{d-2}\chi_k(\eta\cdot)$ instead. Then χ^η solves:

$$\begin{cases} -\mathscr{L}^{\lambda,\mu}[\chi_k^\eta](x) = \eta^d e_k, & x \in \eta^{-1}\mathbb{T}^d \setminus T, \\ \chi_k^\eta(x) = 0, & x \in \partial T. \end{cases}$$

» The oscillating test functions will be built from the further rescaled function $v_k^\varepsilon = \chi_k^\eta(\frac{x}{\varepsilon\eta})$ ($d \geq 3$) and $v^\varepsilon = |\log \eta|^{-1}\chi_k^\eta(\frac{x}{\varepsilon\eta})$ (for $d = 2$), which satisfies

$$\begin{cases} -\mathscr{L}^{\lambda,\mu}[v_k^\varepsilon](x) = \dfrac{1}{\sigma_\varepsilon^2} e_k, & x \in \varepsilon\mathbb{R}^d_{f,\eta}, \\ v_k^\varepsilon(x) = 0, & x \in \cup_{\ell\in\mathbb{Z}^d}\, \varepsilon(\ell + \eta T). \end{cases}$$

» We need to find limits for quantities involving v_k^ε, in various regimes, and quantitatively.

Quantitative results

Theorem (J. '21)

For the dilute regimes, assume $u \in W^{2,d} \cap H_0^1(\Omega)$ *for* $d \geq 3$ *and* $u \in W^{2,\infty} \cap H_0^1(\Omega)$ *for* $d = 2$*. Then:*

(i) In super-critical setting,

$$\|\frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} - f^k(x)v_k^\varepsilon(x)\|_{H^1(\Omega)} + \frac{1}{\sigma_\varepsilon}\|\frac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} - f^k(x)v_k^\varepsilon(x)\|_{L^2(\Omega)} \leq C(\sigma_\varepsilon + \eta^{\frac{d-2}{2}})\|f\|_{W^{2,d}}$$

(ii) In the critical setting, and $\sigma_\varepsilon \to \sigma_0$*,*

$$\|\tilde{u}_\varepsilon - \sigma_\varepsilon^2(\frac{M}{\sigma_0^2}u)^k v_k^\varepsilon\|_{H^1(\Omega)} \leq C(\varepsilon + |\sigma_\varepsilon^2 - \sigma_0^2|)\|u\|_{W^{2,d}}.$$

(iii) In the sub-critical setting,

$$\|\tilde{u}_\varepsilon - (Mu)^k v_k^\varepsilon\|_{H^1(\Omega)} \leq C(\sigma_\varepsilon^{-2} + \eta^{\frac{d-2}{2}})\|u\|_{W^{2,d}}.$$

For $d = 2$*, the above results hold with* $W^{2,d}$ *replaced by* $W^{2,\infty}$*, and* $\eta^{\frac{d-2}{2}}$ *replaced by* $|\log \eta|^{-\frac{1}{2}}$*.*

Corrector results

» One can read correctors from the quantitative estimates.

» Take $d \geq 3$ and the sub-critical setting for example. Write

$$\tilde{u}_\varepsilon - (Mu)^k v_k^\varepsilon = \tilde{u}_\varepsilon - u - r^\varepsilon, \qquad r^\varepsilon := (Mu)^k \left[v_k^\varepsilon - M^{-1} e_k \right].$$

» r^ε serves as corrector, because $u_\varepsilon - (u + r^\varepsilon)$ has much better convergence.

» We have estimates for $\|v_k^\varepsilon - M^{-1} e_k\|$, hence

$$\|\tilde{u}_\varepsilon - u\|_{L^{\frac{2d}{d-2}}} \leq C\eta^{\frac{d-2}{2}} \|u\|_{W^{2,d}}$$

Remarks

» The case of "clustering" is OK.

» The setting of Neumann boundary data can be studied, *very* different:

- Usually can only assume zero data in $\partial\Omega^{\varepsilon}_{\text{int}}$.
- In that case, dilute ($\eta \to 0$) v.s. non-dilute ($\eta \sim 1$), and $\overline{A}(\eta) \to I$.

» A drawback: convergence rates in the super-critical setting is non-optimal.

» Really go to multi-scale ?

» J. (2021), "*Convergence rate for the homogenization of stationary diffusions in dilutely perforated domains with reflecting boundaries*".

Uniform gap in spectra

» A uniform gap of the spectrum of NP operator with respect to ε:

Theorem (Fu-J. 2022)

There exists $\delta > 0$ *such that for any* $0 < \varepsilon < 1$ *the spectrum of the NP operator*

$$\mathbb{K}_{D_\varepsilon}^{\lambda,\mu,*} : H_{\mathbf{R}}^{-\frac{1}{2}}(\partial D_\varepsilon) \to H_{\mathbf{R}}^{-\frac{1}{2}}(\partial D_\varepsilon)$$

is contained in $(-\frac{1}{2} + \delta, \frac{1}{2} - \delta)$.

» Self-adjoint w.r.t. proper inner product

» min-max formula reduces to energy arguments

» Ideas from Bonnetier *et. al.*