Strong symmetry breaking rhythms created by folded singularities in pairs of coupled, identical, fast-slow oscillators

Nazir Awal, Irv Epstein, Tasso Kaper, Theo Vo

(Brandeis, Brandeis, Boston, Monash)

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Symmetry breaking

- elementary particle physics
- coupled chemical oscillators
- networks of phase oscillators
- chimeras
- pattern formation
- synaptically coupled neurons, ...

Symmetry-breaking arises when systems have certain symmetries, but interesting attractors (or states) do not share these.

Many known symmetry breaking states are close to symmetric states, such as in-phase or anti-phase

Strong symmetry breaking: substantially different amplitudes or qualitatively different types of oscillations.

Awal, Bullara, Epstein Chaos 2019; Awal and Epstein PRE 2020; Awal and Epstein PRE 2021.
Awal, Epstein, Kaper, Vo: Journal of Nonlinear Science 2024 (April) volume 34, article 53

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OUTLINE

Coupled, identical, fast-slow oscillators

3 examples: Lengyel-Epstein oscillators; van der Pol oscillators; Koper oscillators

Classical in-phase and anti-phase rhythms (symmetric states)

Strong symmetry breaking rhythms: SAO-LAO

Key folded nodes and folded saddles: ““traffic officers”

Explosion of strong symmetry breaking limit cycle canards (SAO-LCC)

Strong symmetry breaking rhythms: SAO-MMO, MMO-MMO, LAO-MMO

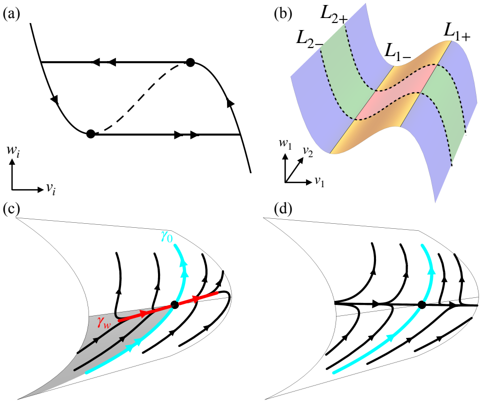
Key folded nodes and folded saddles: “traffic officers”

Conclusions and current work

Abbreviations: SAO = small-amplitude oscillation, LAO = large-amplitude oscillation, MMO = mixed-mode oscillation.

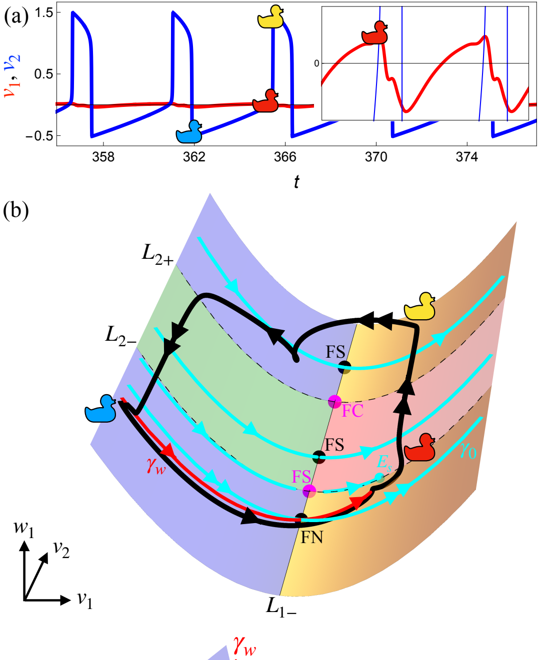
Coupled identical van der Pol relaxation oscillators

$$\dot{v}_1 = \alpha v_1 - h v_1^3 - w_1,$$
$$\dot{w}_1 = \varepsilon\left(v_1 - \theta + d(w_2 - w_1)\right),$$
$$\dot{v}_2 = \alpha v_2 - h v_2^3 - w_2,$$
$$\dot{w}_2 = \varepsilon\left(v_2 - \theta - d(w_2 - w_1)\right).$$



9 patches on critical manifold; folded singularities are on $L_{1,\pm}$, $L_{2,\pm}$.

Strong symmetry breaking SAO-LAO rhythms (vdP)



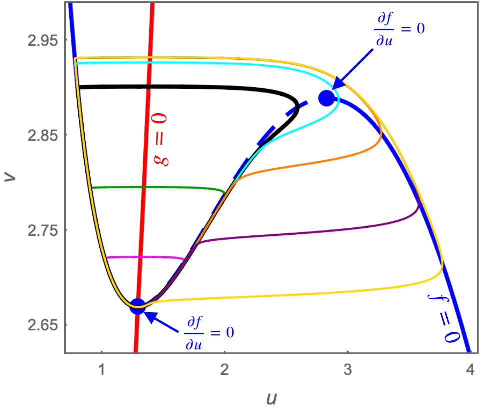
The Lengyel-Epstein oscillator (CIMA and CDIMA)

$$\dot{u} = a - u - \frac{4uv}{1+u^2}$$

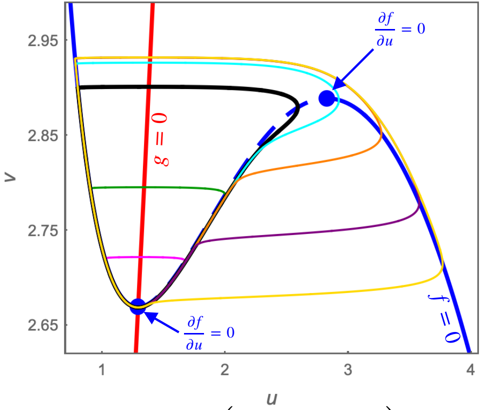
$$\dot{v} = \beta\left(u - \frac{uv}{1+u^2}\right)$$

u fast activator, v slow inhibitor

$a > 3\sqrt{3} \approx 5.19615$, $0 < \beta \ll 1$.



The Lengyel-Epstein oscillator (CIMA and CDIMA)



Canard point at $(u_c, v_c, a_c) = \left(\sqrt{\frac{5}{3}}, \frac{8}{3}, 5\sqrt{\frac{5}{3}}\right)$ (singular limit)

$a_c(\beta) = 5\sqrt{\frac{5}{3}} + 5\beta + \mathcal{O}(\beta^2)$ for $0 < \beta \ll 1$

Limit Cycle Canards: transition f. p. $\rightarrow$ relaxation oscillation

(Note: $5\sqrt{\frac{5}{3}} = 6.454972\ldots$)

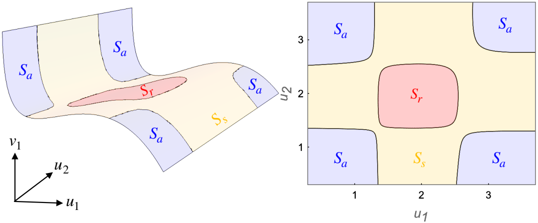
Coupled identical Lengyel-Epstein oscillators

$$\dot{u}_1 = a - u_1 - \frac{4u_1 v_1}{1 + u_1^2} + d_u(u_2 - u_1),$$

$$\dot{v}_1 = \beta\left(u_1 - \frac{u_1 v_1}{1 + u_1^2} + d_v(v_2 - v_1)\right),$$

$$\dot{u}_2 = a - u_2 - \frac{4u_2 v_2}{1 + u_2^2} - d_u(u_2 - u_1),$$

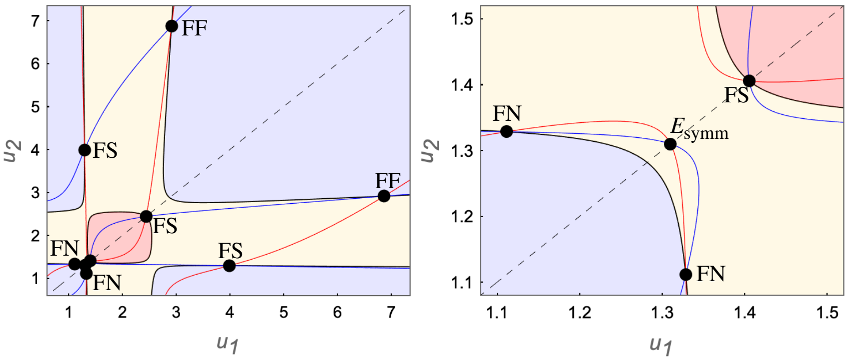
$$\dot{v}_2 = \beta\left(u_2 - \frac{u_2 v_2}{1 + u_2^2} - d_v(v_2 - v_1)\right).$$



6 patches on the critical manifold: attracting S_a, repelling S_r, saddle S_s.

Folded singularities on the fold curves.

Folded singularities of coupled LE oscillators



Folded singularities for $a = 6.55, d_u = 0.1$, and $d_v = 0.5$.

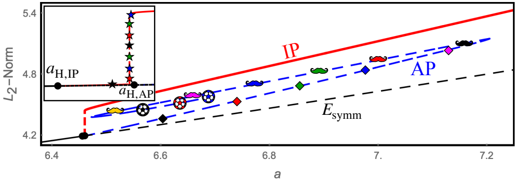
FNs are key traffic officers, and γ_s and γ_w are key lane markings

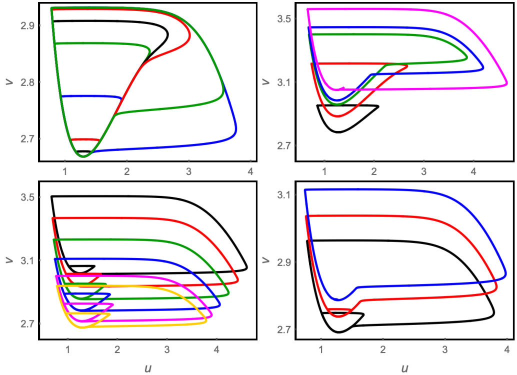
Coupled identical van der Pol oscillators have similar types folded singularities on- and off- the symmetry axis.

Classical in-phase and anti-phase oscillations (LE)

In-phase (IP) rhythms: IP SAO-SAO, IP LCC-LCC, IP LAO-LAO.

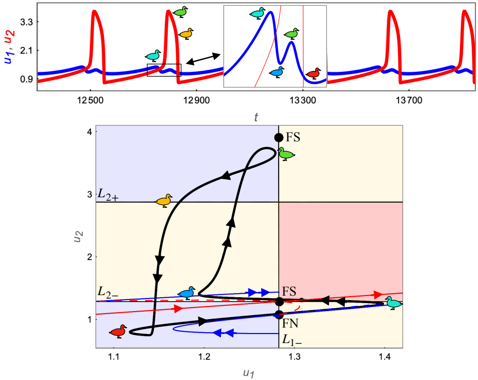
Anti-phase (AP) rhythms: AP SAO-SAO, AP LCC-LCC, AP LAO-LAO.





Strong symmetry breaking SAO-LAO (LE: γ_s)

Oscillator 1: two SAOs, oscillator 2: one LAO, per period.



Orbit (black): close to γ_s (blue, double arrows) of the FN.

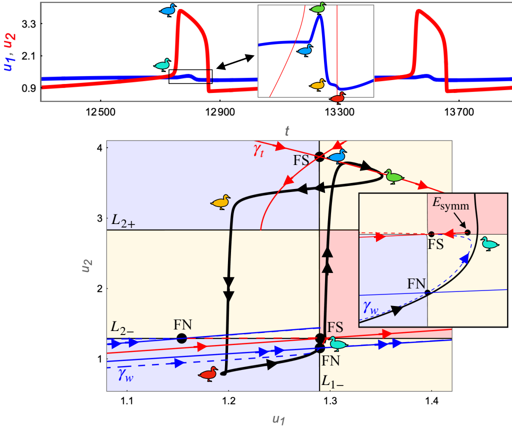
FN = key mechanism responsible for strong symmetry breaking

Oscillator 1 remains near local min, oscillator 2 makes anan LAO

Timing: first SAO of oscillator 1 before the up-jump of oscillator 2

($a = 6.54, d_u = 8 \times 10^{-4}, d_v = 1.0588$ with $\beta = 0.003$)

Strong symmetry breaking SAO-LAO (LE: γ_w)



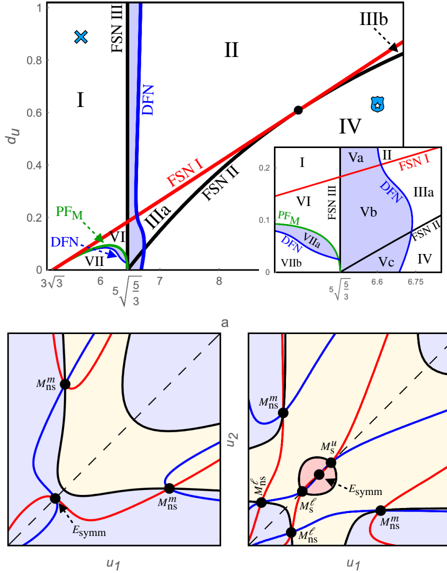
Passage through neighbourhood of FN is near γ_w.

FNs are key traffic officers, and γ_s and γ_w are key lane markings

First maximum in oscillator 1 occurs after the up-jump of osc. 2

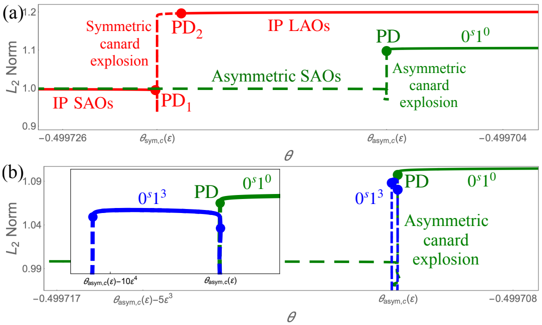
($a = 6.47$, $d_u = 8 \times 10^{-4}$, $d_v = 0.5$, and $\beta = 0.001$)

SAO-LAO exist in regions Va-c, created by FNs (LE)



Bifurcation diagram for folded singularities (coupled LE)

How are SAO-LAOs created? Through an explosion of symmetry breaking SAO-LCC rhythms! (vdP)



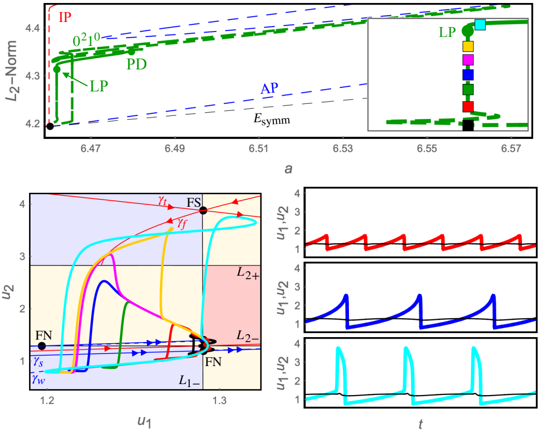
SAO-LCCs: transition between asymmetric SAO-SAO attractors and strong symmetry breaking SAO-LAO attractors

Multi-stability with In-phase (IP) solutions

Strong symmetry breaking SAO-MMO attractors (0^s1^3)

($\varepsilon = 0.01, \alpha = 1.5, h = 2,$ and $d = 0.0137638.$)

Explosion of strong symmetry breaking SAO-LCC rhythms (LE)



Primary asymmetric canard explosion: creates 0^21^0 SAO-LAO rhythms.

($d_u = 8 \times 10^{-4}$, $d_v = 0.5$, and $\beta = 0.001$.)

Geometric desingularization: SAO-LCC explosions

One oscillator undergoes an explosion of limit cycle canards (LCCs) while the other exhibits SAOs.

van der Pol:

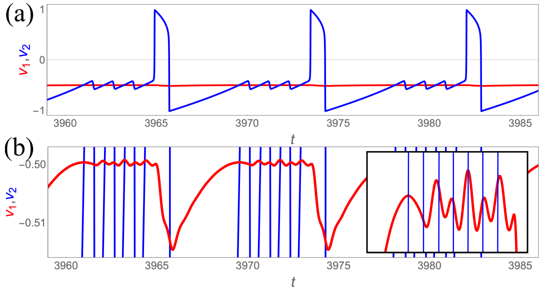
$$\theta_{\mathrm{asym,c}}(\varepsilon) = -\sqrt{\frac{\alpha}{3h}} + \varepsilon\left(\frac{1}{8\alpha\sqrt{3\alpha h}} + \frac{d}{4\sqrt{3\alpha h}}\right) + \mathcal{O}(\varepsilon^2).$$

LE:

$$a_{\mathrm{asym,c}}(\beta) = 5\sqrt{\frac{5}{3}} + 5\beta + \frac{4}{3}\sqrt{\frac{5}{3}}d_u + \frac{2}{3}\sqrt{\frac{5}{3}}\beta d_v + \frac{153\sqrt{15}}{64}\beta^2 + \mathcal{O}(\beta^3).$$

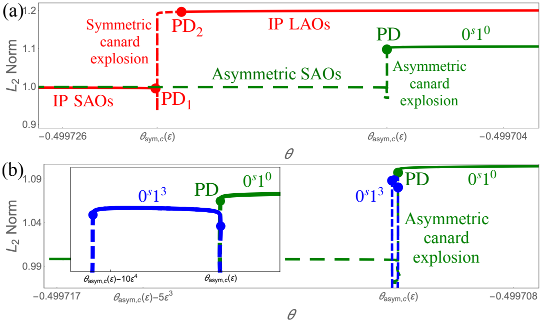
What else? Strong symmetry breaking SAO-MMOs

$0^6 1^3$ SAO-MMO attractors (coupled, identical van der Pol oscillators)



The FN on L_{1-} is again the key singularity, and the trajectory is in $\ell = 3$ rotational sector of the FN funnel.
($\varepsilon = 0.01, \alpha = 1.5, h = 2, d = 0.0137638, \theta = -0.4997099.$)

Strong symmetry breaking SAO-MMO (vdP)

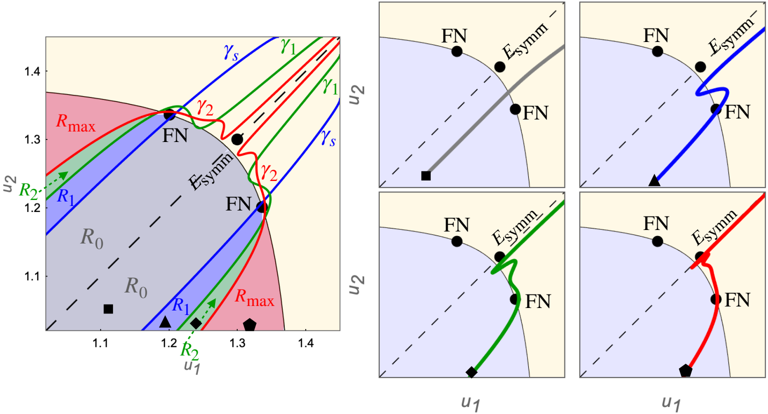


Strong symmetry breaking SAO-MMO attractors (0^s1^3)

Exist near the asymmetric explosion of SAO-LCCs

($\varepsilon = 0.01, \alpha = 1.5, h = 2,$ and $d = 0.0137638.$)

Secondary canards and rotation sectors of FNs (LE)

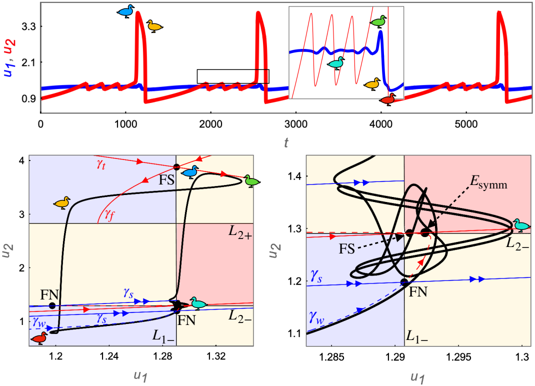


Example: the non-symmetric FNs have eigenvalue ratio $\mu \approx 0.187$. Hence, for each FN, also exist two secondary canards.

($a = 6.5, d_u = 0.15, d_v = 0.5$, and $\beta = 0.001$.)

Coupled identical van der Pol oscillators also have two such key FNs, and depending on parameters these can have secondary canards.

Strong symmetry breaking SAO-MMO rhythms (LE)

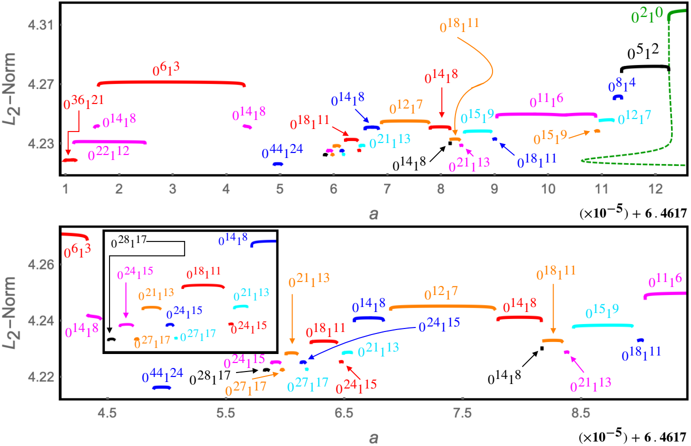


Strong symmetry breaking $0^6 1^3$ SAO-MMO rhythm.

The orbit has a segment near the true canard (blue to green ducky); there, oscillator 1 attains the maximum of its largest SAO.

$(a = 6.46174, d_u = 8 \times 10^{-4}, d_v = 0.5,\ \beta = 0.001.\ )$

Strong symmetry breaking SAO-MMO rhythms (LE)



A veritable menagerie of SAO-MMO rhythms in an $\mathcal{O}\left(\beta^{3/2}\right)$ neighbourhood of the leftmost asymmetric canard explosion, $a_{\mathrm{asym,c}}(\beta)$

($d_u = 8 \times 10^{-4}$, $d_v = 0.5$, and $\beta = 0.001$.)

Symmetrically-coupled, identical Koper oscillators

$$\begin{aligned}
\varepsilon\dot{x}_1 &= y_1 - f_1(x_1) + d_x(x_2 - x_1)\\
\dot{y}_1 &= g_1(x_1, y_1, z_1) + d_y(y_2 - y_1)\\
\dot{z}_1 &= h_1(x_1, y_1, z_1) + d_z(z_2 - z_1)\\
\varepsilon\dot{x}_2 &= y_2 - f_2(x_2) - d_x(x_2 - x_1)\\
\dot{y}_2 &= g_2(x_2, y_2, z_2) - d_y(y_2 - y_1)\\
\dot{z}_2 &= h_2(x_2, y_2, z_2) - d_z(z_2 - z_1)
\end{aligned}$$

$f_i(x_i) = x_i^3 - 3x_i$ (cubic fast nullcline)

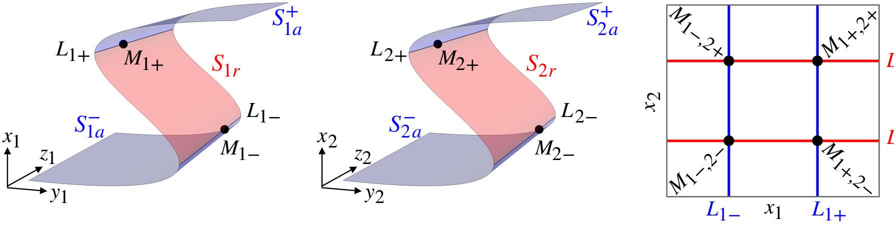
$g_i(x_i, y_i, z_i) = kx_i - 2(y_i + \lambda) + z_i$

$h_i(x_i, y_i, z_i) = \lambda + y_i - z_i$ for $i = 1, 2$.

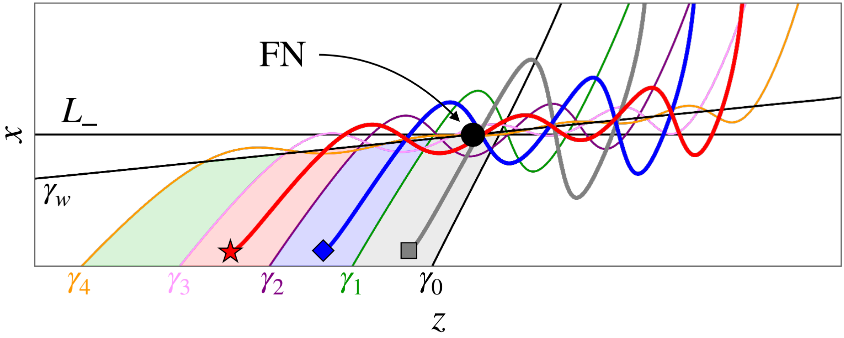
Parameters: $0 < \varepsilon \ll 1$, λ, and k identical in both oscillators

Linear, diffusive coupling: d_x, d_y, and d_z non-negative

Coupled Koper oscillators: critical manifold, fold sets, and folded singularities



FN and its rotation sectors (Koper)



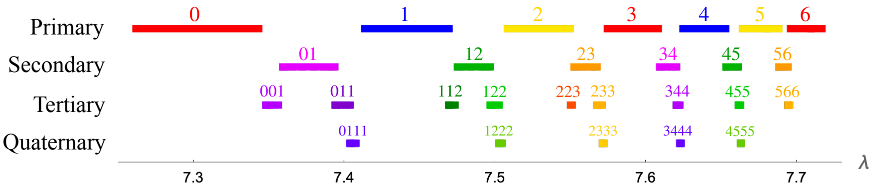
Maximal canards, γ_i for $i = 0, 1, 2, 3, 4$, and w, of a FN.

Sector R_1 (grey shaded) between γ_0 (black) and γ_1 (green):
one SAO (grey curve) about γ_w (black)

Sector R_2 (blue shaded) between γ_1 (green) and γ_2 (purple):
two SAOs (blue curve) about γ_w

R_3 sector (red-shaded) and R_4 sector (green-shaded)

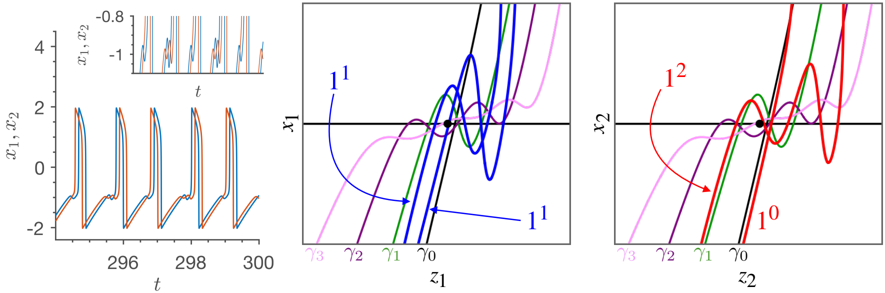
Strong symmetry breaking SAO-MMO rhythms (coupled Koper)



SAO-MMO CLASSIFICATION: Primary-, secondary-, tertiary-, and quaternary SAO-MMO attractors

With 2 slow variables in each Koper oscillator, there are so many more types of strong symmetry breaking rhythms.....

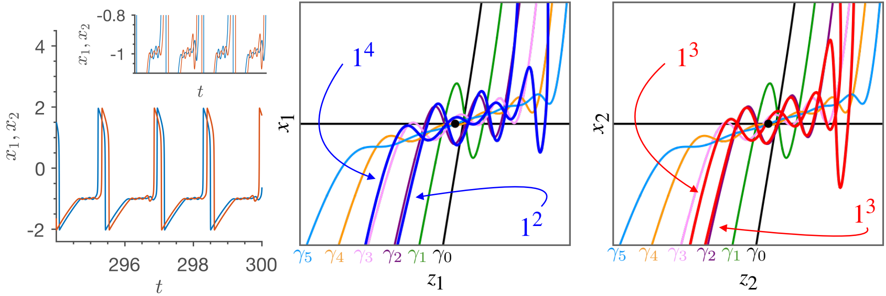
Strong symmetry breaking MMO-MMO attractors (coupled Koper)



$1^1 1^1$–$1^0 1^2$

$\lambda = 6.910$, $k = -10$, $\varepsilon = 0.01$, $d_z = 14$.

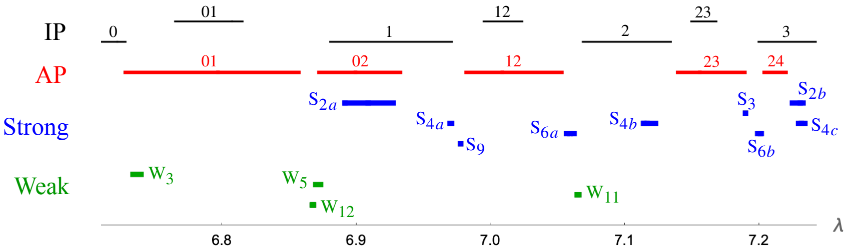
Strong symmetry breaking MMO-MMO rhythms (coupled Koper)



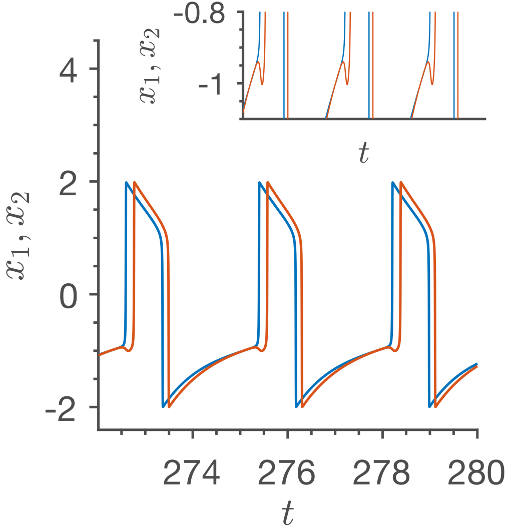
$1^4 1^2$–$1^3 1^3$

$\lambda = 7.219,\ k = -10,\ \varepsilon = 0.01,\ d_z = 14.$

Multi-stability in the coupled Koper oscillators



Strong symmetry breaking LAO-MMO rhythm (coupled Koper)



Conclusions and current work

Coupled, identical relaxatio (fast-slow) oscillators

Strong symmetry breaking SAO-LAO and SAO-MMO rhythms in van der Pol and LE (each osc. has 1 fast and 1 slow variable)

Strong symmetry breaking SAO-LAO, SAO-MMO, MMO-MMO, and LAO-MMO rhythms in Koper oscillators (each osc. has 1 fast and 2 slow variables)

Folded singularities: FNs, double folded nodes, FSN-II points = primary mechanisms creating the strong symmetry breaking

Types and timing of strong symmetry breaking attractors pidetermined by passage near the canards of the folded singularities

Geometric desingularization analysis of the explosion of asymmetric LCCs (limit cycle canards)

Natural extensions to systems with three (or more) coupled identical oscillators

Natural extensions to oscillators with multiple slow and/or fast variables

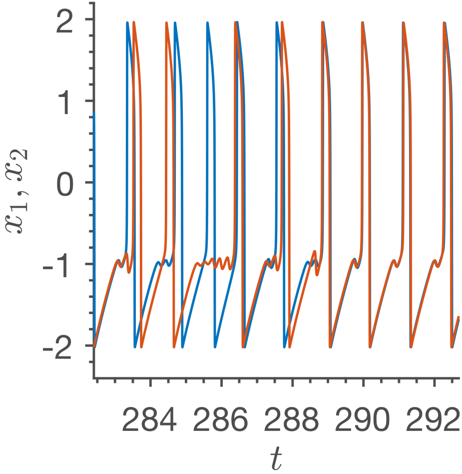
References

Awal, Epstein, Kaper, Vo: Journal of Nonlinear Science 2024 (April) volume 34, article 53

Awal, Epstein, Kaper, Vo: Chaos 2024 (May) volume 34(5)

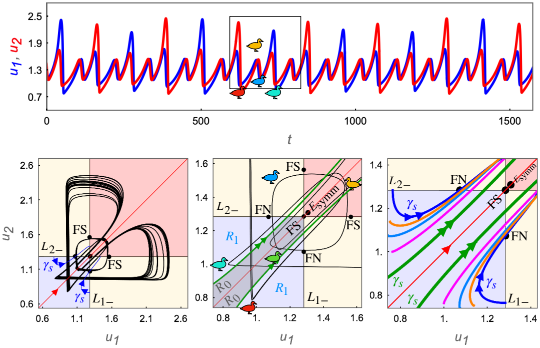
THANK YOU

Spliced MMO-MMO



Strong "splicing" symmetry breaking MMO-MMO rhythms (time-periodic). (a) $\lambda = 7.120$, (b) $\lambda = 7.134$, (c) $\lambda = 6.970$, and (d) $\lambda = 6.973$. (a) A $1^2 1^3 1^1 1^0$-$1^3 1^1 1^6$ rhythm. (b) A $1^2 1^3 1^1 1^0 (1^2 1^3)^3$-$1^3 1^1 1^6 (1^3 1^2)^3$ rhythm, which is more complex

Canardioids (LE)



A weak symmetry breaking canardioid: near a quasiperiodic AP 1^3

($a = 6.54$, $d_u = 8 \times 10^{-4}$, $d_v = 1.0588$, and $\beta = 0.014$.)