Transverse modulational dynamics of quenched patterns

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Abstract and Motivation

We study transverse modulational dynamics of striped pattern formation in the wake of a directional quenching mechanism. Such mechanisms have been proposed to control pattern-forming systems and suppress defect formation in many different physical settings, such as light-sensing reaction-diffusion equations, solidification of alloys, and eutectic lamellar crystal growth. In the context of two prototypical pattern forming PDEs, the complex Ginzburg-Landau and Swift-Hohenberg equations, we show that long-wavelength and slowly varying modulations of striped patterns are governed by a one-dimensional viscous Burgers' equation, with viscous and nonlinear coefficients determined by the quenched stripe selection mechanism.

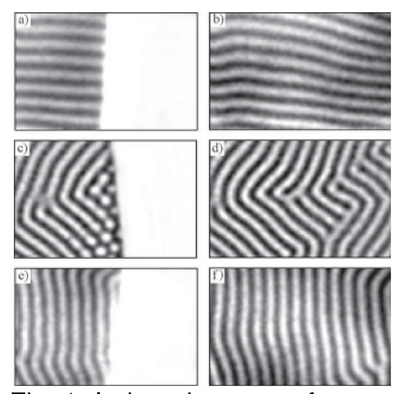

Fig. 1: Induced patterns from the introduction of a traveling mask in light-sensitive CDIMA reaction (Ref. 5).

Complex Ginzburg-Landau Equation

- Quenched Complex Ginzburg-Landau (CGL) equation with $\xi = x - ct$

$$A_t = (1+\mathrm{i}\alpha)(\partial_\xi^2 + \partial_y^2)A + cA_\xi + \chi(\xi)A - (1+\mathrm{i}\gamma)A|A|^2$$

$$A \in \mathbb{C}, \quad (x,y) \in \mathbb{R}^2, \quad \alpha, \gamma \in \mathbb{R}$$

- Quenching mechanism $\chi(\xi) = -\mathrm{sign}(\xi)$, controls stability of $A \equiv 0$.

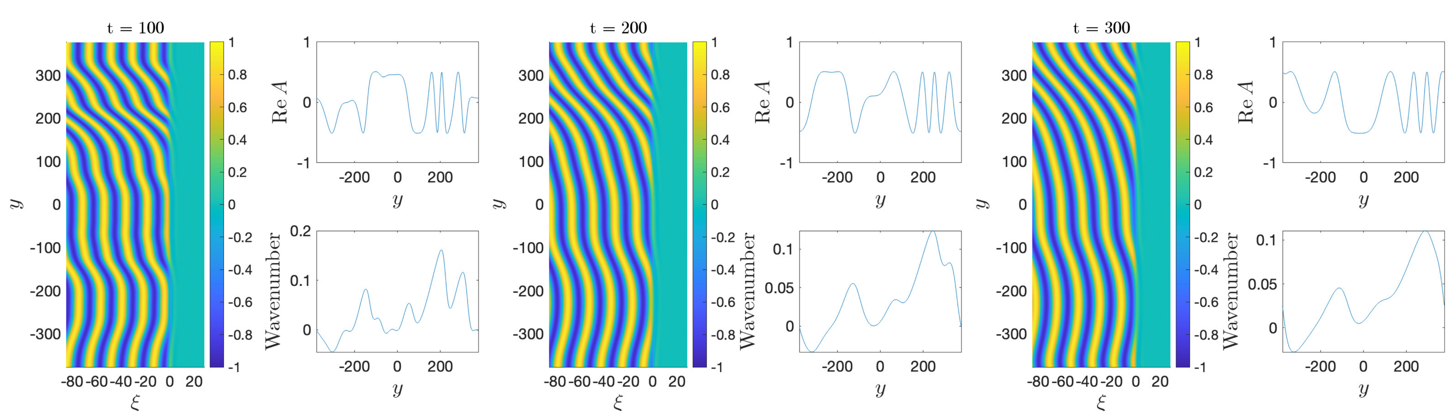

Fig. 2: Evolution of 2D quenched pattern in (ξ, y) variables with $\alpha = 3, \gamma = 1, c = 2.5$, from small random initial data.

- Pure stripes $re^{\mathrm{i}(k_x\xi + k_y y - \omega t)}$, have nonlinear dispersion relation

$$r^2 = 1 - (k_x^2 + k_y^2), \quad \omega(k_x, k_y) = (\alpha - \gamma)(k_x^2 + k_y^2) - ck_x + \gamma$$

- Stripe-forming front solutions

$$A(\xi, y, t) = e^{\mathrm{i}(k_y y - \omega t)} A_\mathrm{f}(\xi; c, k_y)$$

- Travelling wave eqn. with asymptotic boundary conditions

$$0 = (1+\mathrm{i}\alpha)(\partial_\xi^2 - k_y^2)A_\mathrm{f} + cA_{\mathrm{f},\xi} + (\chi(\xi) + \mathrm{i}\omega)A_\mathrm{f} - (1+\mathrm{i}\gamma)A_\mathrm{f}|A_\mathrm{f}|^2$$

$$0 = \lim_{\xi \to -\infty} \left|A_\mathrm{f}(\xi) - re^{\mathrm{i}k_x\xi}\right|, \qquad 0 = \lim_{\xi \to +\infty} A_\mathrm{f}(\xi)$$

- Fronts with $k_y = 0$ exist (Ref. 3) for $c \lesssim 2\sqrt{1+\alpha^2}$.
- $k_y \neq 0$ is a regular perturbation, so fronts generically persist. ω and k_x are selected by c and k_y.

$$k_{x,\mathrm{f}}(k_y; c) = k_{x,\mathrm{f}}(0) + \beta_2 k_y^2 + \mathcal{O}(k_y^4), \qquad k_y \sim 0,$$

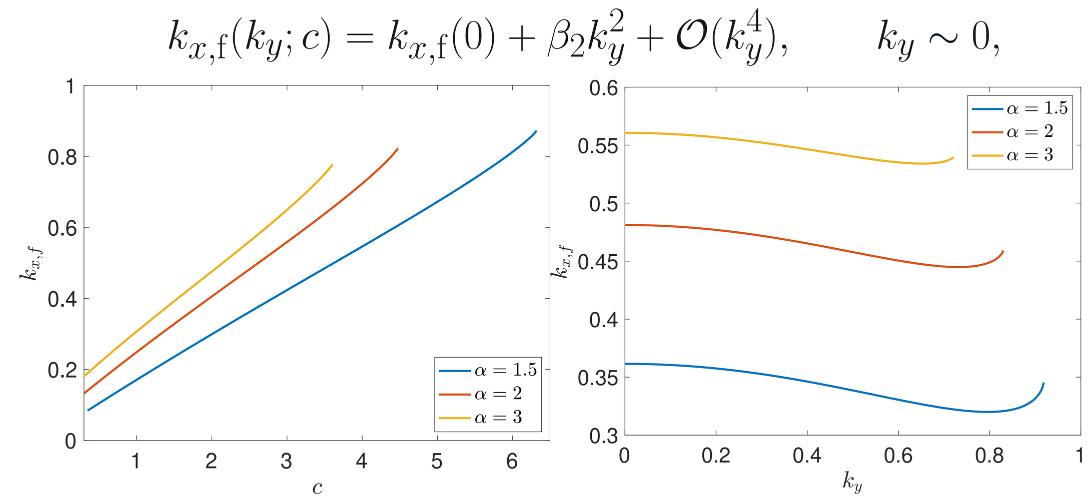

Fig. 3: Wave number selection curves with $\gamma = 1$ and $\alpha = 1.5, 2, 3$.

Slow Transverse Modulations

- Slowly-varying transverse phase modulation function $\Phi(Y,T)$ with slow variables $Y = \delta y$, $T = \delta^2 t$ for some small parameter $0 < \delta \ll 1$.

$$A(\xi, y, t) = e^{\mathrm{i}(\Phi(Y,T) - \omega_\mathrm{f} t)} \left[A_\mathrm{f}(\xi; \delta\Phi_Y(Y,T)) + \delta^2 w_1(\xi, Y, T; \delta)\right].$$

- Fix $\xi = \xi_0$ directly behind quench with $-1 \leq \xi_0 < 0$.
- Expand and collect $\mathcal{O}(\delta^2)$ terms. Solvability condition gives a viscous Burgers' equation for slowly-varying transverse wave number modulation $\Psi := \partial_Y \Phi$

$$\Psi_T = \frac{\lambda''_\mathrm{lin}(0)}{2}\Psi_{YY} + \frac{\omega''_\mathrm{f}(0)}{2}(\Psi^2)_Y$$

- $\lambda''_\mathrm{lin}(0) \approx 2(1 + \alpha\gamma)$ and $\omega''_\mathrm{f}(0)$ given by

$$\partial_{k_y}^2 \omega_\mathrm{f}(0) = 2(\alpha - \gamma) + \partial_{k_y}^2 k_{x,\mathrm{f}}(0)\left(2(\alpha - \gamma)k_{x,\mathrm{f}}(0) - c\right)$$

Example: Source-Sink Transverse Defect Pair

- Small transverse wave number $k_{y,+} = \delta q_+$ for $y > 0$ and $k_{y,-} = \delta q_-$ for $y > 0$, with $0 < \delta \ll 1$ and $q_\pm = \mathcal{O}(1)$.

$$A(\xi, y, 0) = h(-\xi)\Big(h(y)r_- e^{\mathrm{i}(k_{x,-}\xi + k_{y,-}y)} + h(-y)r_+ e^{\mathrm{i}(k_{x,+}\xi + k_{y,+}y)}\Big)$$

- Wave numbers $k_{x,\pm}$ chosen so that $k_{x,\pm} = k_x(k_{y,\pm})$, $r_\pm^2 = \sqrt{1 - (k_{x,\pm}^2 + k_{y,\pm}^2)}$.
- Defect speed determined by $c_\mathrm{d} = c_{g,0} + \delta c_* = \delta \frac{\omega''_\mathrm{f}(0)}{2}(q_- + q_+)$

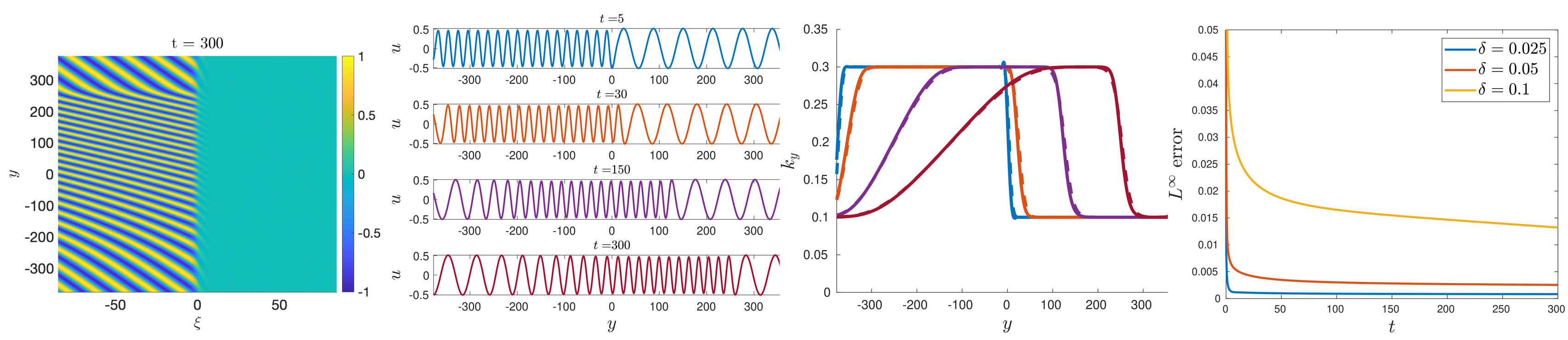

Fig. 4: Source-sink defect pair with $\alpha = 3, \gamma = 1, c = 2.5$ obtained from initial condition which connects stripes solutions with transverse wave numbers $k_{y,-} = 0.3$, $k_{y,+} = 0.1$ and $\delta = 0.1$; Local wave number measured as $\psi(y,t) = \mathrm{Im}\, A_y(\xi_0, y, t)/A(\xi_0, y, t)$.

Example: Phase-Slip Defect Modulation

- Localized defect with $\phi_0(Y) = \pi\,\mathrm{erf}(Y)$ and $\mathrm{erf}(Y) = 2\pi^{-1/2}\int_0^Y e^{-t^2}dt$

$$A(\xi, y, 0) = h(-\xi)\sqrt{1-k^2}\exp\left[\mathrm{i}(k_x\xi + \delta y + \phi_0(\delta y))\right]$$

- Choose transverse wave number $k_y = \delta$ so that

$$\Psi(Y) = 1 + 4\pi^{1/2}e^{-Y^2}$$

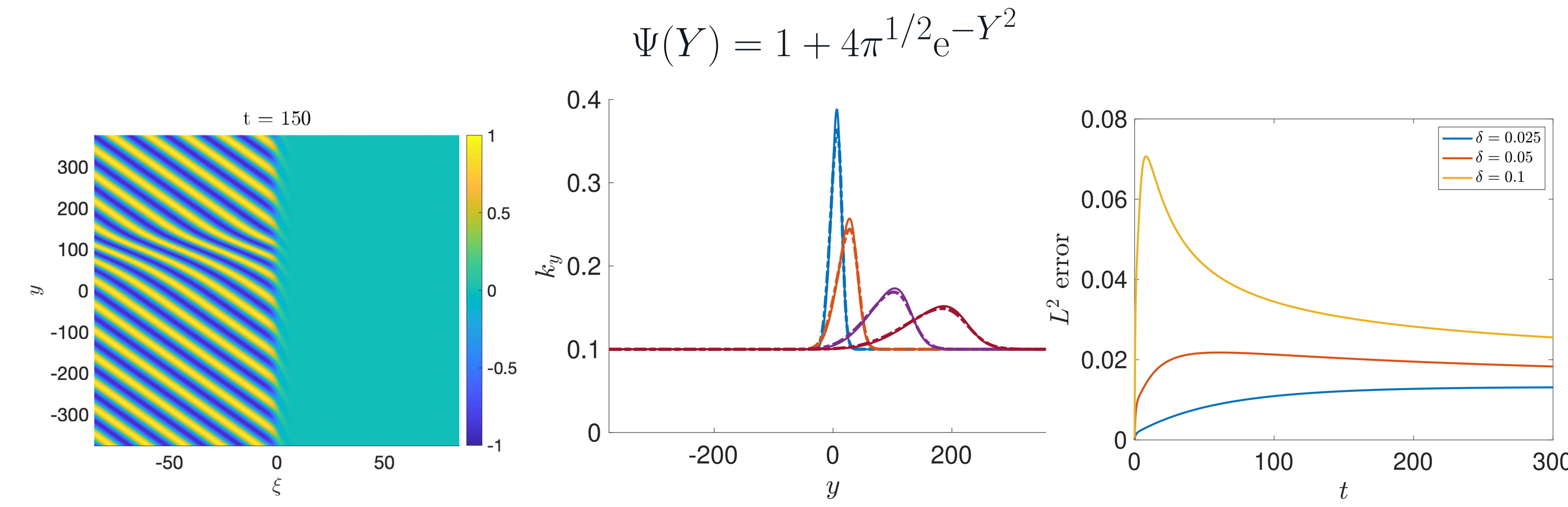

Fig. 5: Top row: Localized phase-slip defect solution of with $c = 2.5, \alpha = 3, \gamma = 1$ and $\delta = 0.1$ as well as L^2 error.

Swift-Hohenberg Equation

- Swift-Hohenberg Equation in co-moving frame

$$u_t = -(1 + \partial_\xi^2 + \partial_y^2)^2 u + c\partial_\xi u + \mu\chi(\xi)u - u^3, \qquad \mu > 0$$

- Front solutions $u_\mathrm{f}(\xi, k_y y + \omega t)$, periodic in $\zeta := k_y y + \omega t$ exist (Ref. 4), solve

$$0 = -(1 + \partial_\xi^2 + k_y^2\partial_\zeta^2)^2 u_\mathrm{f} + (c\partial_\xi - \omega\partial_\zeta)u_\mathrm{f} + \mu\chi(\xi)u_\mathrm{f} - u_\mathrm{f}^3$$

- Striped fronts perturb smoothly for $k_y \sim 0$ and

$$k_{x,\mathrm{f}}(k_y) = k_{x,\mathrm{f}}(0) + \beta_2 k_y^2 + \mathcal{O}(k_y^4). \qquad \beta_2 = \frac{1}{c}\langle 2\partial_\zeta^2(1+\partial_\xi^2)u_\mathrm{f}(\cdot,\cdot;0), e_*\rangle_{L^2_\eta}$$

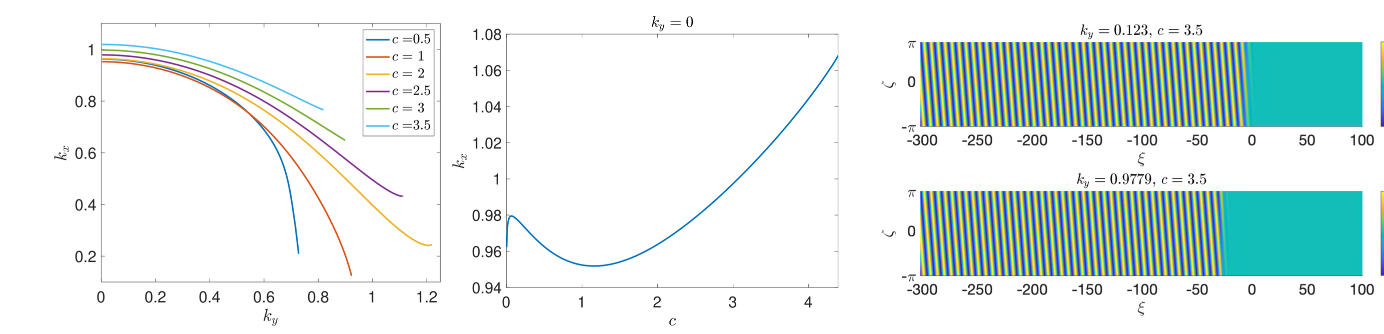

Fig. 6: Swift-Hohenberg wave number selection curves for $\mu = 1$, depicting $k_{x,}(c, k_y)$ for c fixed, and k_y varied, then $k_y = 0$ fixed and c varied; Example front profiles $u_\mathrm{f}(\xi, \zeta; k_y)$ with $c = 3.5$ and $k_y = 0.123, 0.9779$; $\mu = 1$.

- Bloch wave theory and concavity of k_y in k_x imply

$$\lambda''_\mathrm{lin}(0) = \langle 4(1 + (k_x\partial_z)^2)\partial_z u_\mathrm{f}, b_*\rangle_{L^2(\mathbb{T}_{2\pi})}, \qquad \omega''_\mathrm{f}(0) = 2\beta_2 c$$

- Phase-slip initial condition $u(\xi, y, 0) = \sqrt{4\mu/3}\cos(k_x x + k_y y + \phi_0(\delta y))h(-\xi)$
- Numerical wave number measurements from Iterative Hilbert Transform; Gibbs-type oscillations in shocks

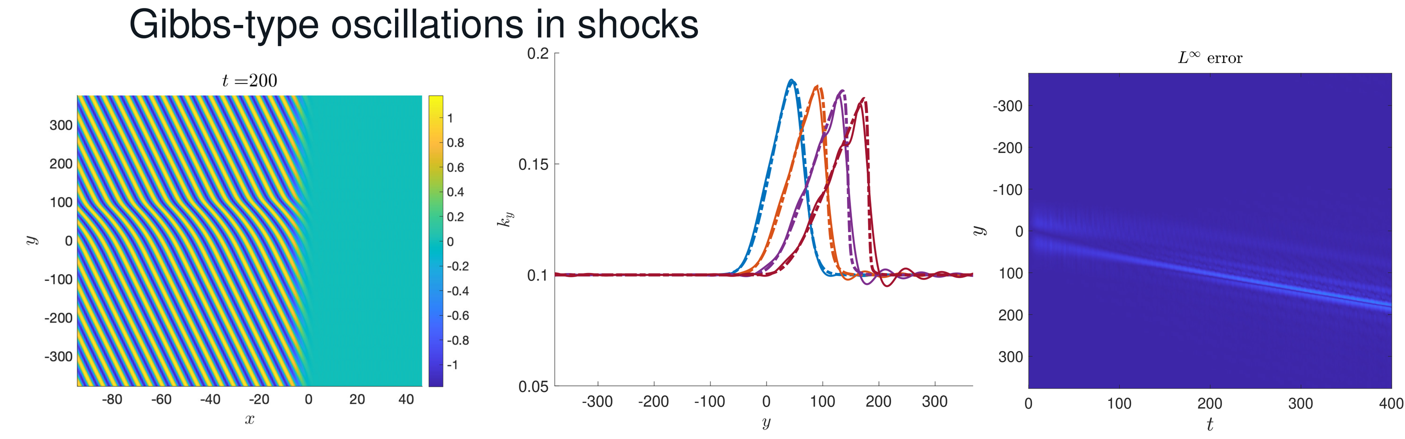

Fig. 7: Localized phase perturbation of striped front with $(k_x, k_y) = (0.993, 0.1)$; $c = 3, \mu = 1$; Transverse wave number dynamics at $\xi = \xi_0 = -1$ fixed (solid) plotted against rescaled viscous Burgers' solution at times $t = 100$(blue), 200(orange), 300(purple), 400(red); plot of absolute error.

Future Work

- Far-field dynamics and behavior
- Rigorous approximation arguments
- Effect of domain geometry on pattern growth

References & Acknowledgements

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