Existence of transit orbits in the planar restricted 3-body problem via variational methods

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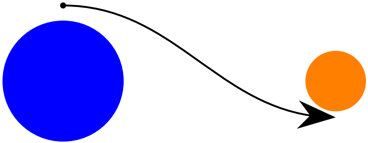
Relation to spacecraft trajectory design

This research is related to the spacecraft trajectory design, for example, to realize **a mission that sends a spacecraft from the vicinity of the Earth to the vicinity of the Moon**.



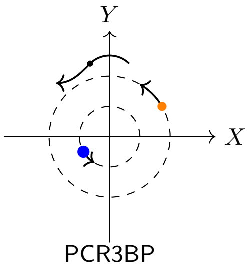
In the theoretical field of spacecraft trajectory design,

- in the past, even for this case, using **the two-body problem** of a celestial body and a spacecraft had been common.
- recently, using **the three (or more)-body problem** has been studied actively. For this case, **the restricted three-body problem** is appropriate.

In designing the trajectory, it is hoped that the mission can be accomplished **in a short time** and **at a low cost**. In this case, to design the trajectory **at a low cost**, it is necessary to consider whether **a transit orbit** exists or not.

We give **sufficient conditions** in a form verifiable by numerical calculations for the existence of **transit orbits** in **the restricted three-body problem** via **variational methods**.

- **The restricted three-body problem** (**R3BP**) is a mathematical model that describes the motion of an object that **is affected by the gravitation force from two celestial bodies**, but whose **effect on them is negligible**.
- Since the plane identical to the two bodies is an invariant plane, we can consider the problem that the motion is restricted on this plane (**PR3BP**).
- The two bodies are assumed to be in **a bounded motion** of the two-body problem, i.e., they are in **a circular motion** (**PCR3BP**) or **a elliptical motion** (**PER3BP**).



Lagrangian for PCR3BP・PER3BP

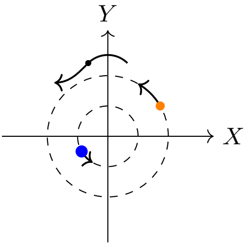
PCR3BP・PER3BP

$$L_\mu^C(\boldsymbol{z},\boldsymbol{v}) := \frac{1}{2}|\boldsymbol{v}|^2 + {}^t\boldsymbol{z}J\boldsymbol{v} + V_\mu(\boldsymbol{z}) \qquad : \text{PCR3BP (Autonomous)}$$

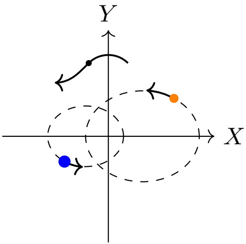
$$L_\mu^E(\boldsymbol{z},\boldsymbol{v},t) := \frac{1}{2}|\boldsymbol{v}|^2 + {}^t\boldsymbol{z}J\boldsymbol{v} + f(t)V_\mu(\boldsymbol{z}) \qquad : \text{PER3BP (Non-Autonomous)}$$

$0 < \mu \leq \frac{1}{2}$ (Mass ratio of two bodies), $V_\mu(\boldsymbol{z}) := \frac{1}{2}|\boldsymbol{z}|^2 + \frac{1-\mu}{|\boldsymbol{z} + {}^t(\mu,0)|} + \frac{\mu}{|\boldsymbol{z} - {}^t(1-\mu,0)|} > 0$

$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $0 \leq e < 1$ (Eccentricity of elliptical orbit), $f(t) := \frac{1}{1 + e\cos t}$



PCR3BP



PER3BP

PCR3BP

E-L eq.:

$$\begin{aligned}\dot{\boldsymbol{z}} &= \boldsymbol{v},\\ \dot{\boldsymbol{v}} &= 2J\boldsymbol{v} + \nabla V_\mu(\boldsymbol{z}).\end{aligned}$$

Lagrange point (i.e. the equilibrium point) $(\boldsymbol{z}_0, \boldsymbol{v}_0)$:

$$\nabla V_\mu(\boldsymbol{z}_0) = \mathbf{0},\ \ \boldsymbol{v}_0 = \mathbf{0}.$$

PER3BP

E-L eq.:

$$\begin{aligned}\dot{\boldsymbol{z}} &= \boldsymbol{v},\\ \dot{\boldsymbol{v}} &= 2J\boldsymbol{v} + f(t)\nabla V_\mu(\boldsymbol{z}).\end{aligned}$$

Lagrange point (i.e. the equilibrium point) $(\boldsymbol{z}_0, \boldsymbol{v}_0)$:

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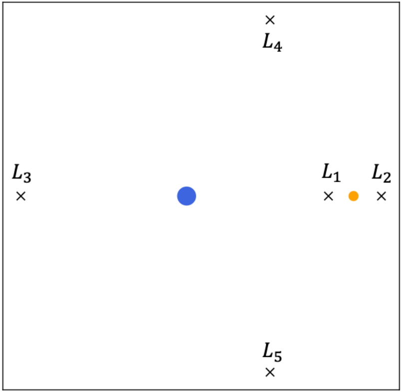
Lagrange points for PCR3BP • PER3BP



Energy for PCR3BP・PER3BP

PCR3BP

Energy function:

$$E_\mu^C(\boldsymbol{z}, \boldsymbol{v}) := \frac{1}{2}|\boldsymbol{v}|^2 - V_\mu(\boldsymbol{z}).$$

For the solution of PCR3BP $c^T : [0, T] \to \mathbb{R}^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_\mu^C(c^T(t), \frac{\mathrm{d}c^T}{\mathrm{d}t}(t))\right) = 0.$$

So, energy is preserved.

PER3BP

Energy function:

$$E_\mu^E(\boldsymbol{z}, \boldsymbol{v}, t) := \frac{1}{2}|\boldsymbol{v}|^2 - f(t)V_\mu(\boldsymbol{z})$$

For the solution of PER3BP $c^T : [0, T] \to \mathbb{R}^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_\mu^E(c^T(t), \frac{\mathrm{d}c^T}{\mathrm{d}t}(t), t)\right) = \frac{\partial E_\mu^E}{\partial t}(c^T(t), \frac{\mathrm{d}c^T}{\mathrm{d}t}(t), t) = -\frac{\mathrm{d}f}{\mathrm{d}t}(t)\, V_\mu(c^T(t)).$$

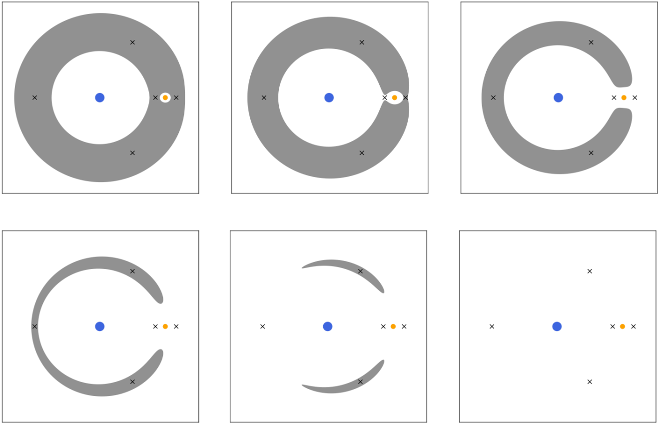
So, energy is not preserved. $\left(\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \frac{e\sin t}{(1+e\cos t)^2},\ V_\mu > 0\right)$

Hill Regions for PCR3BP

Hill Regions for PCR3BP

$$\{\boldsymbol{z} \in \mathbb{R}^2 \mid \exists \boldsymbol{v} \in \mathbb{R}^2 \text{ s.t. } E^C_\mu(\boldsymbol{z}, \boldsymbol{v}) = \frac{1}{2}|\boldsymbol{v}|^2 - V_\mu(\boldsymbol{z}) = h\}$$

$$=\{\boldsymbol{z} \in \mathbb{R}^2 \mid h + V_\mu(\boldsymbol{z}) \geq 0\}$$



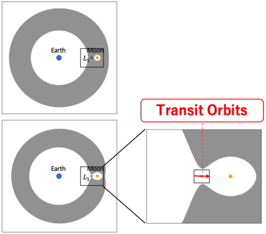
Transit orbits of PCR3BP

Hill Region

$$\{\boldsymbol{z} \mid h + V_\mu(\boldsymbol{z}) > 0\}$$

$h < E_\mu^C(L_1)\big(= -V_\mu(L_1)\big)$

$h > E_\mu^C(L_1)\big(= -V_\mu(L_1)\big)$



Transit orbits of PCR3BP at L_1

- **Start from** the region to **the left of** L_1.
- **Reach** the region to **the right of** L_1.
- With **energy values** such that the Hill region has **the "tunnel"**.

The transit orbit of PCR3BP can be constructed by the solution of a two-point boundary value problem (2PBVP) with fixed energy condition.

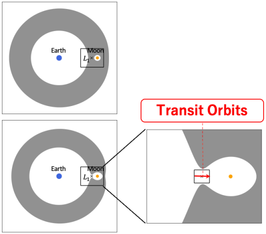
Transit orbits of PER3BP

Hill Region

$$\left\{ \boldsymbol{z} \;\middle|\; h + \min_{t\in[0,2\pi)} f(t)V_\mu(\boldsymbol{z}) > 0 \right\}$$

$$h < \max_{t\in[0,2\pi)} X_\mu^E(L_1, t)$$
$$\left(= -\min_{t\in[0,2\pi)} f(t)V_\mu(L_1)\right)$$

$$h > \max_{t\in[0,2\pi)} X_\mu^E(L_1, t)$$
$$\left(= -\min_{t\in[0,2\pi)} f(t)V_\mu(L_1)\right)$$



Transit orbits of PER3BP at L_1

- **Start from** the region to **the left of** L_1.
- **Reach** the region to **the right of** L_1.
- With **energy values** such that the Hill region has **the "tunnel"** at a certain time.

The transit orbit of PER3BP can be constructed by the solution of 2PBVP with fixed endpoint's energy condition.

Minimizing methods for 2PBVP (PCR3BP・PER3BP)

For

$$\text{a compact convex set } R \subset \Omega \ (:= \mathbb{R}^2 \setminus \{{}^t(-\mu, 0), {}^t(1-\mu, 0)\}),$$
$$\boldsymbol{z}_S, \boldsymbol{z}_G \in R \text{ and } T > 0,$$

we define the set of Sobolev curves in R with 2PBV $\boldsymbol{z}_S, \boldsymbol{z}_G$ and time T

$$\mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G, T; R) := \left\{ c^T \in H^{1,2}((0,T), \mathbb{R}^2) \;\middle|\; \begin{array}{l} c^T(0) = \boldsymbol{z}_S, \ c^T(T) = \boldsymbol{z}_G, \\ c^T([0,T]) \subset R \end{array} \right\}.$$

And we define the action functional of PCR3BP (resp. PER3BP)

$$\mathcal{A} : \mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G, T; R) \to \mathbb{R}, \ \mathcal{A}(c^T) := \int_{[0,T]} L_\mu^C(c^T(t), \dot{c}^T(t)) \text{ (resp. } L_\mu^E) \, \mathrm{d}t$$

Theorem

For any $R, \boldsymbol{z}_S, \boldsymbol{z}_G$ and T satisfying the above conditions,
there exists a minimizer c_*^T of $\mathcal{A}$ in $\mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G, T; R)$.

$$\text{i.e. } \exists c_*^T \in \mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G, T; R) \text{ s.t. } \mathcal{A}(c_*^T) = \inf\{\mathcal{A}(c^T) \mid c^T \in \mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G, T; R)\}.$$

Furthermore, if $c_*^T(0,T) \subset \mathrm{int}(R)$ holds, then c_*^T is the classical solution of PCR3BP (resp. PER3BP).

Minimizing methods for 2PBVP with fixed energy condition (PCR3BP)

For

$h \in \mathbb{R}$, a compact convex set $R \subset \{\boldsymbol{z} \in \Omega \mid \forall \boldsymbol{v} \in \mathbb{R}^2, L_\mu^C(\boldsymbol{z}, \boldsymbol{v}) + h > 0\}$,
and $\boldsymbol{z}_S, \boldsymbol{z}_G \in R$ ($z_S \neq z_G$)

we define the set of Sobolev curves in R with 2PBV $\boldsymbol{z}_S, \boldsymbol{z}_G$

$$\mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G; R) := \bigcup \{\mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G, T; R) \mid T > 0\}$$

And we define the action functional of PCR3BP with energy

$$\mathcal{A}_h : \mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G; R) \to \mathbb{R},\ \mathcal{A}_h(c^T) := \int_{[0,T]} L_\mu^C(c(t), \dot{c}(t)) + h \,\mathrm{d}t$$

Main Lemma 1 (PCR3BP)

For any h, R and $\boldsymbol{z}_S, \boldsymbol{z}_G$ satisfying the above conditions,
there exists a time-free minimizer $c_*^{T_*}$ of $\mathcal{A}_h$ in $\mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G; R)$.

i.e. $\exists c_*^{T_*} \in \mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G; R)$ s.t. $\mathcal{A}_h(c_*^{T_*}) = \inf\{\mathcal{A}_h(c) \mid c \in \mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G; R)\}$.

Furthermore, if $c_*^{T_*}(0, T_*) \subset \mathrm{int}(R)$ holds, then $c_*^{T_*}$ is the classical solution of PCR3BP with energy value h.

Minimizing methods for 2PBVP with fixed endpoint's energy condition (PER3BP)

For

$h \in \mathbb{R}$, a compact convex set $R \subset \{\boldsymbol{z} \in \Omega \mid \forall t \in \mathbb{R}, \boldsymbol{v} \in \mathbb{R}^2, L_\mu^E(\boldsymbol{z}, \boldsymbol{v}, t) + h > 0\}$,
and $\boldsymbol{z}_S, \boldsymbol{z}_G \in R$ $(z_S \neq z_G)$

we define the set of Sobolev curves in R with 2PBV $\boldsymbol{z}_S, \boldsymbol{z}_G$

$$\mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G; R) := \bigcup \{\mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G, T; R) \mid T > 0\}$$

And we define the action functional of PER3BP with energy

$$\mathcal{A}_h : \mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G; R) \to \mathbb{R},\ \mathcal{A}_h(c^T) := \int_{[0,T]} L_\mu^E(c(t), \dot{c}(t), t) + h \ \mathrm{d}t$$

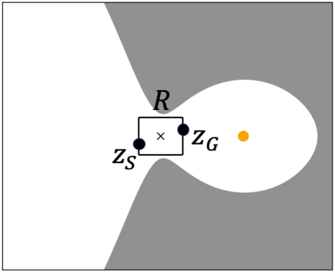
Main Lemma 2 (PER3BP)

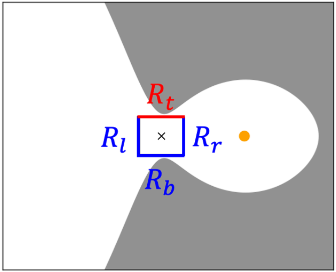
For any h, R and $\boldsymbol{z}_S, \boldsymbol{z}_G$ satisfying the above conditions,
there exists a time-free minimizer $c_*^{T_*}$ of $\mathcal{A}_h$ in $\mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G; R)$.

i.e. $\exists c_*^{T_*} \in \mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G; R)$ s.t. $\mathcal{A}_h(c_*^{T_*}) = \inf\{\mathcal{A}_h(c) \mid c \in \mathcal{C}(\boldsymbol{z}_S, \boldsymbol{z}_G; R)\}$.

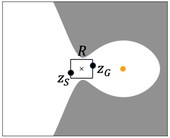
Furthermore, if $c_*^{T_*}(0, T_*) \subset \mathrm{int}(R)$ holds, then $c_*^{T_*}$ is the classical solution of PER3BP with endpoint's energy value h.

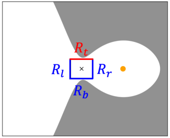
How to choose a curve set





Main Result 1・2



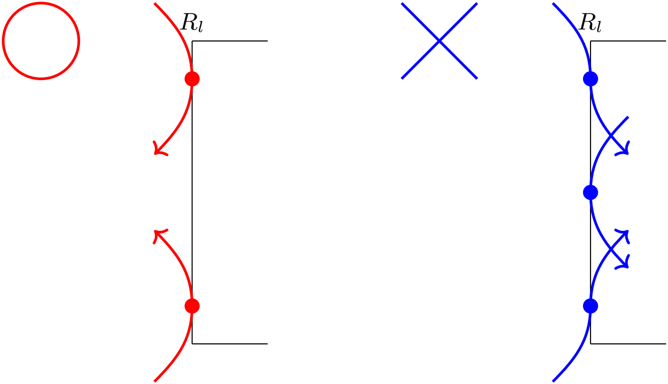


	PCR3BP(Main Result 1)	PER3BP(Main Result 2)
Cond.1	"Positivity of integrand" ⇒ There exists a time-free minimizer.	"Positivity of integrand" ⇒ There exists a time-free minimizer.
Cond.2	"External contact condition on $R_{l,r}$ for the solution of energy h" ⇒ Time-free minimizer does not reach $R_{l,r}$.	"External contact condition on $R_{l,r}$ for the solution of endpoint's energy h" ⇒ Time-free minimizer does not reach $R_{l,r}$.
Cond.3	"Directed (left-to-right) external contact condition on R_b for the solution of energy h" ⇒ Time-free minimizer does not reach R_b.	"Directed (left-to-right) external contact condition on R_b for the solution of endpoint's energy h" ⇒ Time-free minimizer does not reach R_b.
Cond.4	"If the curve reaches R_t, then it is not a time-free minimizer." ⇒ Time-free minimizer does not reach R_t.	"If the curve reaches R_t, then it is not a time-free minimizer." ⇒ Time-free minimizer does not reach R_t.

External contact condition on R_l

For any solution $(x(t), y(t))$ of energy (resp. endpoint's energy) h,

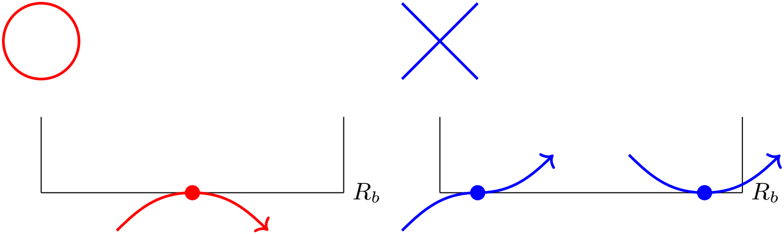
$$(x(t), y(t)) \in R_l,\ \dot{x}(t) = 0 \ \Rightarrow\ \ddot{x}(t) < 0.$$



Directed (left-to-right) External contact condition on R_b

For any solution $(x(t), y(t))$ of energy (resp. endpoint's energy) h,

$$(x(t), y(t)) \in R_b,\ \dot{x}(t) > 0,\ \dot{y}(t) = 0 \ \Rightarrow\ \ddot{y}(t) < 0.$$



Previous research and main results

Although many numerical studies suggest the existence of transit orbits, few mathematical results show their existence.

Mathematical results can be divided into two categories: "those based on the perturbation theory method" and "those based on the variational method". The former cannot verify their existence for concrete situations, but the latter is important in that it allows us to do so.

	Perturbation theory	Variational method
PCR3BP	Moser(1958,[1])	Moeckel(2005,[2]) · Main Result 1
PER3BP	Fitzgerald&Ross(2022,[3])	Main Result 2

[1] J. Moser, On the generalization of a theorem of Liapunov, Communications on Pure and Applied Mathematics, 11:257-271, 1958.

[2] R. Moeckel, A variational proof of existence of transit orbits in the restricted three-body problem, Dynamical Systems, 20:45-58, 2005.

[3] J. Fitzgerald and S. Ross, Geometry of transit orbits in the periodically-perturbed restricted three-body problem, Advances in Space Research, 70:144-156, 2022.

Application to an equal mass system ($\mu = \frac{1}{2}$) (numerical results)

$$L_1 = (0,0),\ E^C_{\frac{1}{2}}(L_1) = -2,\ \max_{t\in[0,2\pi)} E^E_{\frac{1}{2}}(L_1) \approx -1.9998.$$

PCR3BP

Energy h:

$$h = -1.85\ (> -2).$$

Rectangular region centered on L_1: $R = [-c,c] \times [-k,k] \subset \mathbb{R}^2$:

$$c = 0.072,\ k = 0.134.$$

Cond.1-3 hold and Cond.4 is satisfied

by taking $\boldsymbol{z_S}$ to the lower left of R and $\boldsymbol{z_G}$ to the lower right.

PER3BP

Eccentricity e:

$$e = 2.0 \times 10^{-4}.$$

Energy h:

$$h = -1.85\ (> -1.9998).$$

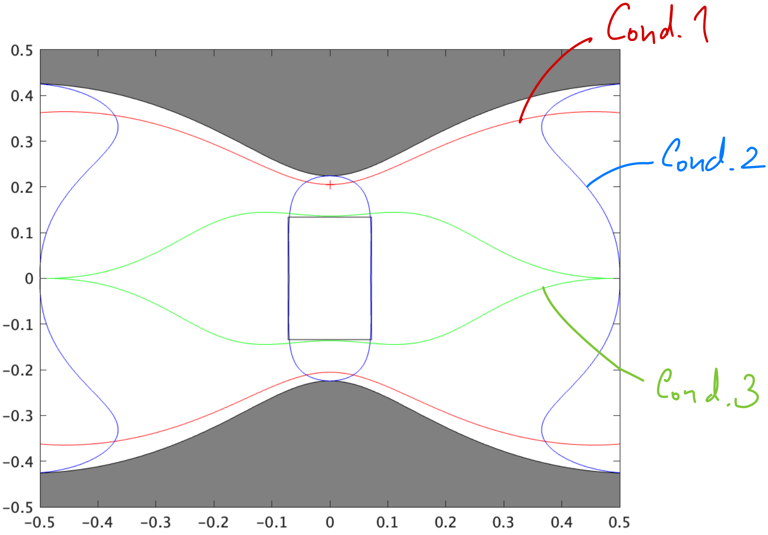
Rectangular region centered on L_1: $R = [-c,c] \times [-k,k] \subset \mathbb{R}^2$:

$$c = 0.072,\ k = 0.134.$$

Cond.1-3 hold and Cond.4 is satisfied

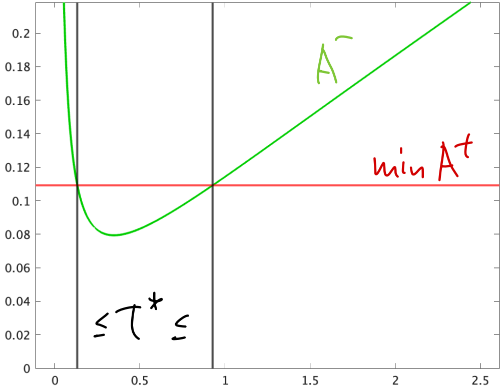
by taking $\boldsymbol{z_S}$ to the lower left of R and $\boldsymbol{z_G}$ to the lower right.

Application to an equal mass PCR3BP ($\mu = \frac{1}{2}$) (numerical results)
(I) R that satisfies Cond.1-3



Application to an equal mass PCR3BP ($\mu = \frac{1}{2}$) (numerical results)
(II) Evaluating the time T_* of the time-free minimizer

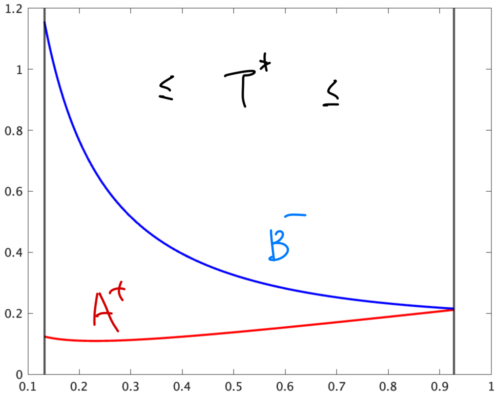
$$A^-(T) \leq \min\{\int_c L_\mu^C + h \;\mathrm{d}t \;\mid\; c \in H^1((0,T);\mathbb{R}^2), \begin{cases} c(0,T) \subset R, \\ c(0) = \boldsymbol{z}_S,\; c(T) = \boldsymbol{z}_G \end{cases} \} \leq A^+(T)$$



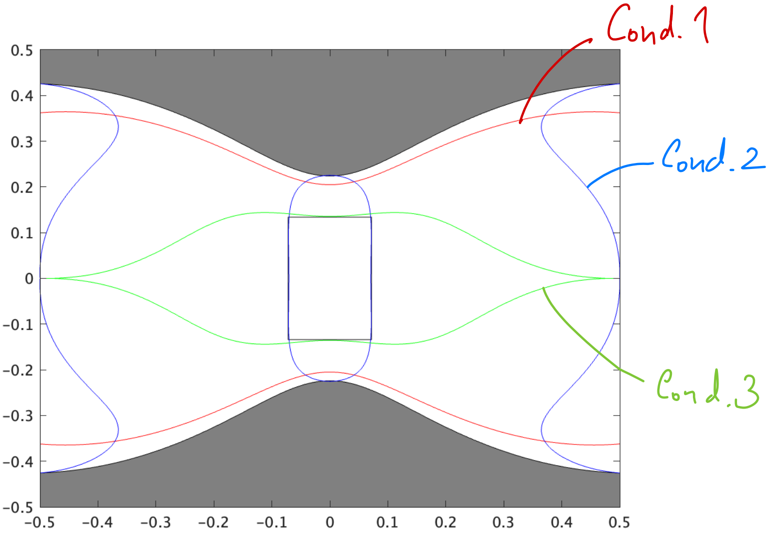
Application to an equal mass PCR3BP ($\mu = \frac{1}{2}$) (numerical results) (III) Verification of Cond.4

$$\min\{\int_c L_\mu^C + h \ \mathrm{d}t \ \mid \ c \in H^1((0,T);\mathbb{R}^2), \begin{cases} c(0,T) \subset R, \\ c(0) = \boldsymbol{z}_S, \ c(T) = \boldsymbol{z}_G \end{cases} \} \leq A^+(T)$$

$$B^-(T) \leq \min\{\int_c L_\mu^C + h \ \mathrm{d}t \ \mid \ c \in H^1((0,T);\mathbb{R}^2), \begin{cases} c(0,T) \subset R, \\ c(0) = \boldsymbol{z}_S, \ c(T) = \boldsymbol{z}_G \\ \exists t \in (0,T) \text{ s.t. } c(t) \in R_t \end{cases} \}$$

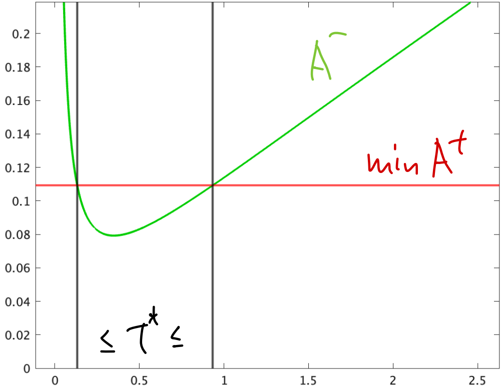


Application to an equal mass PER3BP ($\mu = \frac{1}{2}$) (numerical results)
(I) R that satisfies Cond.1-3



Application to an equal mass PER3BP ($\mu = \frac{1}{2}$) (numerical results)
(II) Evaluating the time T_* of the time-free minimizer

$$A^-(T) \leq \min\{\int_c L_\mu^E + h \; \mathrm{d}t \;\mid\; c \in H^1((0,T);\mathbb{R}^2), \begin{cases} c(0,T) \subset R, \\ c(0) = \boldsymbol{z}_S,\; c(T) = \boldsymbol{z}_G \end{cases} \} \leq A^+(T)$$



Application to an equal mass PER3BP ($\mu = \frac{1}{2}$) (numerical results) (III) Verification of Cond.4

$$\min\{\int_c L_\mu^E + h \ \mathrm{d}t \ \mid \ c \in H^1((0,T);\mathbb{R}^2), \begin{cases} c(0,T) \subset R, \\ c(0) = \boldsymbol{z}_S, \ c(T) = \boldsymbol{z}_G \end{cases} \} \le A^+(T)$$

$$B^-(T) \le \min\{\int_c L_\mu^E + h \ \mathrm{d}t \ \mid \ c \in H^1((0,T);\mathbb{R}^2), \begin{cases} c(0,T) \subset R, \\ c(0) = \boldsymbol{z}_S, \ c(T) = \boldsymbol{z}_G \\ \exists t \in (0,T) \text{ s.t. } c(t) \in R_t \end{cases} \}$$