Transversal family of non-autonomous conformal iterated function systems

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Itereated function system

- **Iterated function system** $\{\phi_1, ..., \phi_k\}$: a collection of uniformly contracting maps on a non-empty compact subset $X \subset \mathbb{R}^m$.

Example

$$\phi_1(x) = \frac{x}{3}, \phi_2(x) = \frac{x}{3} + \frac{2}{3} \text{ on } [0, 1].$$

Then

$$|\phi_i(x) - \phi_i(y)| = \frac{|x - y|}{3}$$

for $i = 1, 2$ and for all $x, y \in [0, 1]$.

Itereated function system

- **Iterated function system**
  $\{\phi_1, \ldots, \phi_k\}$: a collection of uniformly contracting maps on a non-empty compact subset $X \subset \mathbb{R}^m$.
-
$$X_1 := \bigcup_{i=1}^{k} \phi_i(X),$$

$$X_2 := \bigcup_{i=1}^{k} \phi_i(X_1)$$

$$, \ldots,$$

$$X_n := \bigcup_{i=1}^{k} \phi_i(X_{n-1}).$$

  The limit set of the IFS $\{\phi_1, \ldots, \phi_k\}$ is defined by

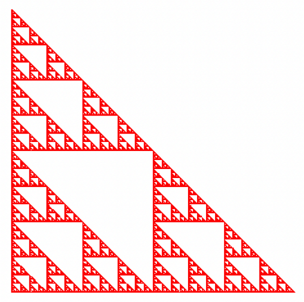
$$\bigcap_{n=1}^{\infty} X_n.$$

Example

$\{\phi_1(x) := x/3, \phi_2(x) := x/3 + 2/3\}$

Middle third Cantor set

Iterated Function Systems



Limit set of an IFS(Sierpiński gasket)

- The limit set A of an IFS $\{\phi_1, \ldots, \phi_k\}$ satisfies the self-similarity;

$$A = \bigcup_{i=1}^{k} \phi_i(A).$$

Hausdorff dimension

Let $A \subset \mathbb{R}^m$. $|A| := \sup\{|x-y| : x, y \in A\}$. For each $t \geq 0$ and $\delta > 0$,

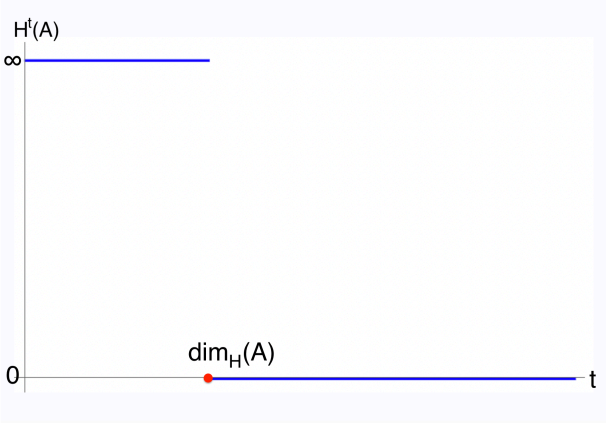
$$\mathcal{H}^t_\delta(A) := \inf\left\{ \sum_{i=1}^{\infty} |U_i|^t : A \subset \bigcup_{i=1}^{\infty} U_i,\ |U_i| \leq \delta\ ,\ U_i \subset \mathbb{R}^m \right\}.$$

$$\mathcal{H}^t(A) := \lim_{\delta \to 0} \mathcal{H}^t_\delta(A) \in [0, \infty].$$

Example

$\dim_H(A) := \sup\{t \geq 0 : \mathcal{H}^t(A) = \infty\} = \inf\{t \geq 0 : \mathcal{H}^t(A) = 0\}$,

called the Hausdorff dimension of A.



Dimension formula

Bowen's formula

Let $\{\phi_1, \ldots, \phi_k\}$ be an iterated function system on a non-empty compact subset $X \subset \mathbb{R}^m$ and let A be the limit set of $\{\phi_1, \ldots, \phi_k\}$. Suppose that ϕ_i is a similitude, i.e. $\phi_i(x) = a_i B_i x + c_i$, where $a_i \in (0, 1), c_i \in \mathbb{R}^m$ and B_i is an orthogonal matrix for any i. If for any $i \neq j \in \{1, \ldots, n\}$

$$\phi_i(X) \cap \phi_j(X) = \emptyset,$$

then we have

$$\dim_H(A) = s,$$

where s is given by

$$\sum_{i=1}^{k} a_i^s = 1.$$

Here, the value s is called the similarity dimension of $\{\phi_1, \ldots, \phi_k\}$.

Example

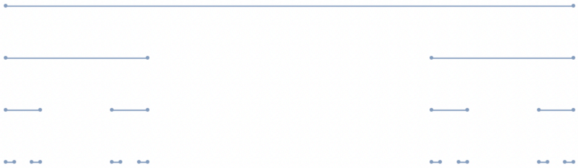
$\{\phi_1(x) := x/3, \phi_2(x) := x/3 + 2/3\}$

Middle third Cantor set A

- $$\left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s = 1 \Rightarrow s = \frac{\log 2}{\log 3}$$
  $$\Rightarrow \dim_H(A) = \frac{\log 2}{\log 3}.$$

$\lambda-$ Cantor set

$$\{\phi_{\lambda,1}(x) := \lambda x, \phi_{\lambda,2}(x) := \lambda x + 1\}(0 < \lambda < 1)$$



$C(\lambda)$

- $$\dim_H(C(\lambda)) = \frac{\log 2}{-\log \lambda}(0 < \lambda < 1/2)$$
  $$\dim_H(C(\lambda)) = 1(1/2 \leq \lambda)$$

Some variation of $\lambda-$ Cantor set

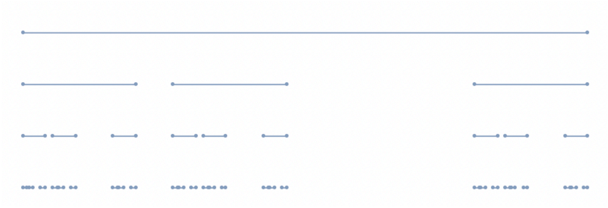
$\{\phi_{\lambda,1}(x) := \lambda x, \phi_{\lambda,2}(x) := \lambda x + 1, \phi_{\lambda,3}(x) := \lambda x + 2\}$
$(0 < \lambda < 1)$

$C_1(\lambda)$

- $$\dim_H(C_1(\lambda)) = \frac{\log 3}{-\log \lambda}(0 < \lambda < 1/3)$$
  $$\dim_H(C_1(\lambda)) = 1(1/3 \leq \lambda)$$

Some variation of λ− Cantor set

$\{\phi_{\lambda,1}(x) := \lambda x, \phi_{\lambda,2}(x) := \lambda x + 1, \phi_{\lambda,3}(x) := \lambda x + 3\}$
$(0 < \lambda < 1/3)$



$C_2(\lambda)$

- $$\dim_H(C_2(\lambda)) = \frac{\log 3}{-\log \lambda} \ (0 < \lambda \leq 1/4)$$
- $$\dim_H(C_2(\lambda)) =? \ (1/4 < \lambda < 1/3)$$

0, 1, 3—problem

Recall

- $$\dim_H(C_2(\lambda)) = \frac{\log 3}{-\log \lambda} \; (0 < \lambda \leq 1/4)$$
- $$\dim_H(C_2(\lambda)) =? \; (1/4 < \lambda < 1/3)$$

Question(Keane 1993)

Is the function $\lambda \mapsto \dim_H(C_2(\lambda))$ continuous on $[1/4, 1/3]$?

Theorem(Pollicott and Simon 1995)

The answer is NO.

- $$\dim_H(C_2(\lambda)) = \frac{\log 3}{-\log \lambda}$$
  for $\mathcal{L}_1-$a.e. $\lambda \in [1/4, 1/3]$.
- There is a dense subset of $[1/4, 1/3]$ on which
  $$\dim_H(C_2(\lambda)) < \frac{\log 3}{-\log \lambda}$$

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Address map

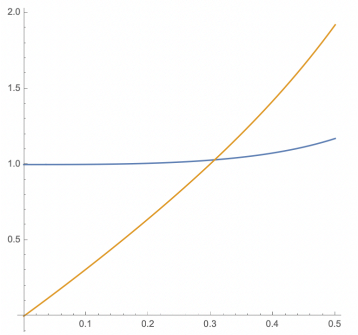
$\{\phi_0(x) := x/3, \phi_1(x) := x/3 + 2/3\}$:IFS, A : limit set.

Define the address map $\pi : \{0,1\}^{\mathbb{N}} \to A$ as follows:

Note that $\pi(\{0,1\}^{\mathbb{N}}) = A$.

Transversality condition

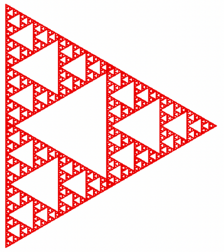
- Φ_λ : IFS, π_λ : Address map, $\lambda \in (0,1)$.
- For any infinite words $\omega \neq \tau$, the transversality condition ensures that $\lambda \mapsto \pi_\lambda(\omega)$ intersect with $\lambda \mapsto \pi_\lambda(\tau)$ transversality:



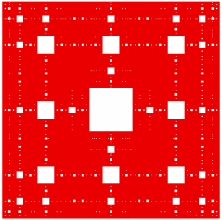
The graphs of $\lambda \mapsto \pi_\lambda(\omega)$ and $\lambda \mapsto \pi_\lambda(\tau)$

Self-similar sets with overlaps

Jordan 2005 investigated "Fat" Sierpiński gaskets, Jordan and Pollicott 2006 investigated "Fat" Sierpiński carpets.



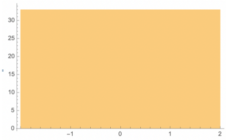
One of the "fat" SGs



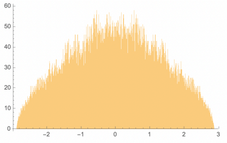
One of the "fat" SCs

Absolute continuity of Bernoulli convolutions

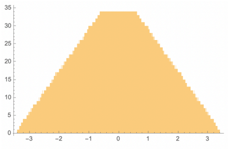
- Solomyak 1995 proved the Bernoulli convolution ν_λ is absolutely continuous for a.e. $\lambda \in [1/2, 1)$ by using the transversality of a power series (Erdős problem). Here, ν_λ is the distribution of $\sum_{n=0}^{\infty} \pm\lambda^n$ where the signs are chosen independently with probability 1/2.



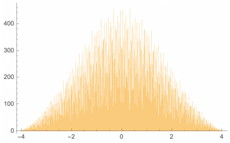
$\nu_{0.5}$



$\nu_{0.651822}$



$\nu_{0.707107}$



$\nu_{0.754877}$

Transversal family

- **Transversal family of IFS**
  Simon, Solomyak and Urbański 2001
- **Transversal family of hyperbolic skew-products**
  Mihailescu and Urbański 2008
- **Transversal family of expanding rational semigroups**
  Sumi and Urbański 2013
- **Transversal family of IFS with inverse**
  Takahashi 2022
- **Transversal family of NIFS** this talk

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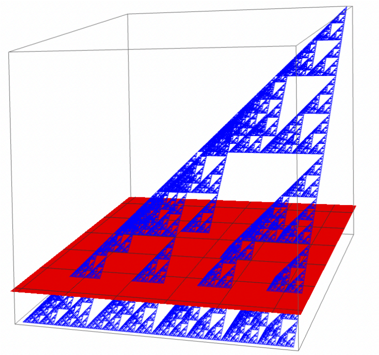
Non-autonomous itereated function system

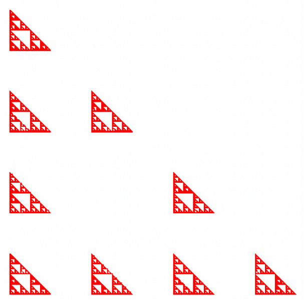
- **Non-autonomous iterated function system** $(\{\phi_1^{(j)}, \ldots, \phi_{k_j}^{(j)}\})_{j=1}^{\infty}$: a sequence of collections of uniformly contracting maps on a non-empty compact subset $X \subset \mathbb{R}^m$.
- $$X_1 := \bigcup_{i=1}^{k_1} \phi_i^{(1)}(X),$$
  $$X_2 := \bigcup_{i=1}^{k_2} \phi_i^{(2)}(X_1)$$
  $$, \ldots,$$
  $$X_n := \bigcup_{i=1}^{k_n} \phi_i^{(n)}(X_{n-1}).$$

  The limit set of a NIFS $(\{\phi_1^{(j)}, \ldots, \phi_{k_j}^{(j)}\})_{j=1}^{\infty}$ is defined by
  $$\bigcap_{n=1}^{\infty} X_n.$$

Slicing 3D Sierpiński gasket



3D Sierpiński gasket and a surface



Limit set of a NIFS as a slice

- Yuto Nakajima, Dimensions of slices through the Sierpiński gasket, Journal of Difference Equations and Applications, Vol.28, No.3, pp. 429-456, 2022.

Bowen's formula

Rempe-Gillen and Urbański 2016

$\Phi = (\{\phi_i^{(j)}\}_{i \in I^{(j)}})_{j=1}^{\infty}$: conformal NIFS on a compact subset $X \subset \mathbb{R}^m$. Suppose that Φ satisfies some mild conditions containing the separation condition: for any $j \in \mathbb{N}$ and any $a \neq b \in I^{(j)}$,

$$\phi_a^{(j)}(X) \cap \phi_b^{(j)}(X) = \emptyset.$$

Then $\dim_H(J) = s_B$, where J denotes the limit set of Φ. The value s_B is called the Bowen dimension.

$$\underline{P}(s) := \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in I^n} \left(\sup_{x \in X} |D\phi_\omega(x)| \right)^s$$

$$\underline{s_B} := \sup\{s \geq 0 \ : \ \underline{P}(s) > 0\}.$$

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Definition

Let $X \subset \mathbb{R}^m$ be a non-empty compact convex set. For each $j \in \mathbb{N}$, let $I^{(j)}$ be a finite set. For any $j \in \mathbb{N}$, let $\Phi^{(j)}$ be a collection $\{\phi_i^{(j)} : X \to X\}_{i \in I^{(j)}}$ of maps $\phi_i^{(j)}$ on X. Let $\Phi = (\Phi^{(j)})_{j=1}^{\infty}$. Suppose that the following holds:

(i) There exists an open connected set $V \supset X$ (independent of i, j) such that for any i, j, $\phi_i^{(j)}$ extends to a C^1 conformal map on V such that $\phi_i^{(j)}(V) \subset V$;

(ii) There is a constant $0 < \gamma < 1$ such that for any $j \in \mathbb{N}$, any $i \in I^{(j)}$,

$$|D\phi_i(x)| \leq \gamma$$

for any $x \in X$.

Then we denote a NIFS $\Phi = (\Phi^{(j)})_{j=1}^{\infty}$ satisfying conditions above by $\Phi \in \Gamma(X, V)$.

Let $\Phi, \Psi \in \Gamma(X, V)$ be such that

$$\Phi = (\{\phi_i^{(j)} : X \to X\}_{i \in I^{(j)}})_{j=1}^{\infty}, \Psi = (\{\psi_i^{(j)} : X \to X\}_{i \in I^{(j)}})_{j=1}^{\infty}.$$

Set

$$||\Phi - \Psi||_\infty = \sup_{j \in \mathbb{N}} \max_{i \in I^{(j)}} ||\phi_i^{(j)} - \psi_i^{(j)}||_\infty,$$

$$||D\Phi - D\Psi||_\infty = \sup_{j \in \mathbb{N}} \max_{i \in I^{(j)}} ||D\phi_i^{(j)} - D\psi_i^{(j)}||_\infty,$$

and

$$d_u(\Phi, \Psi) = \max\{||\Phi - \Psi||_\infty, ||D\Phi - D\Psi||_\infty\},$$

where $||\cdot||_\infty$ denotes the supremum norm.
Let $U \subset \mathbb{R}^d$ be an open set. Consider a family of NIFSs

$$\Phi_t = (\{\phi_{i,t}^{(j)} : X \to X\}_{i \in I^{(j)}})_{j=1}^{\infty} \in \Gamma(X, V), t \in U.$$

Assume that the map $U \ni t \mapsto \Phi_t \in \Gamma(X, V)$ is continuous with respect to the distance d_u.

Address map for NIFS

Let

$$\Phi = (\Phi^{(j)})_{j=1}^{\infty}, \{\phi_i^{(j)} : X \to X\}_{i \in I^{(j)}}.$$

Define the n-th address map

$$\pi_n : \prod_{j=n}^{\infty} I^{(j)} \to \mathbb{R}^m$$

for Φ by

$$\pi_n(\omega) = \bigcap_{j=1}^{\infty} \phi_{\omega|_j, t}(X)$$

for $\omega \in \prod_{j=n}^{\infty} I^{(j)}$.

Remark

In the case of usual IFSs, the n-th address maps π_n are independent of n.

Let $\pi_{n,t}$ be the n-th address map for Φ_t. Assume the following:

Transversality condition

For any compact subset $G \subset U$ there exists a sequence of positive constants $\{C_n\}_{n=1}^{\infty}$ with

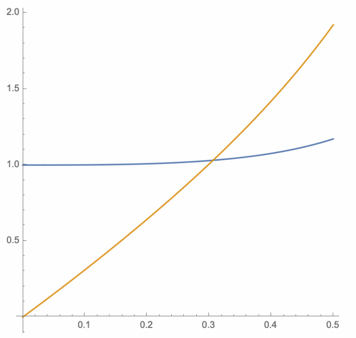
$$\lim_{n\to\infty} \frac{\log C_n}{n} = 0$$

such that for all $\omega, \tau \in I_n^{\infty}$ with $\omega_n \neq \tau_n$ and for all $r > 0$,

$$\mathcal{L}_d\left(\{t \in G \ : \ |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq r\}\right) \leq C_n r^m,$$

where we denote by $\mathcal{L}_d$ the d-dimensional Lebesgue measure on $\mathbb{R}^d$.

If $\{\Phi_t\}_{t\in U}$ satisfies the transversality condition, we say that it is a Transversal family of Non-autonomous Conformal Iterated Function Systems (TNCIFS).



The graphs of $t \mapsto \pi_{n,t}(\omega)$ and $t \mapsto \pi_{n,t}(\tau)$

The lower pressure function

Let $\{\Phi_t\}_{t\in U}$ be a TNCIFS, $\Phi_t = \left(\{\phi_{i,t}^{(j)} : X \to X\}_{i\in I^{(j)}}\right)_{j=1}^{\infty}$.
For each $t \in U$, we define the lower pressure function $\underline{P}_t$.

The lower pressure function

For any $s \geq 0$ and $n \in \{1, 2, ...\}$, we define

$$Z_{n,t}(s) := \sum_{\omega\in I^n} (||D\phi_{\omega,t}||_X)^s,$$

$$\underline{P}_t(s) := \liminf_{n\to\infty} \frac{1}{n} \log Z_{n,t}(s) \in [-\infty, \infty].$$

The function $\underline{P}_t : [0, \infty) \to [-\infty, \infty]$ is called the lower pressure function of Φ_t.

Bowen dimension

Let $t \in U$.

- **Monotonicity:** If $s_1 < s_2$, then either both $\underline{P}_t(s_1)$ and $\underline{P}_t(s_2)$ are equal to ∞, both are equal to $-\infty$, or $\underline{P}_t(s_1) > \underline{P}_t(s_2)$.

$$s(t) := \sup\{s \geq 0 \ : \ \underline{P}_t(s) > 0\} = \inf\{s \geq 0 \ : \ \underline{P}_t(s) < 0\}.$$

The value $s(t)$ is called the Bowen dimension of Φ_t.
For any $t \in U$, let J_t be the limit set of Φ_t.

Remark For any $t \in U$, $\dim_H(J_t) \leq \min\{m, s(t)\}$.

Main Theorem

Main Theorem A

Let $\{\Phi_t\}_{t \in U}$ be a TNCIFS on $\mathbb{R}^m$. Then

$$\dim_H(J_t) = \min\{m, s(t)\} \text{ for } \mathcal{L}^d - \text{a.e. } t \in U;$$

Planar sets with digits tending to the origin

Let $U = B(0, 1/\sqrt{2})\backslash\mathbb{R}$ and $t \in U$. For each $j \in \{1, 2, \ldots\}$, we define $\Phi_t^{(j)} = \{\phi_{1,t}^{(j)}, \phi_{2,t}^{(j)}\} := \{z \mapsto tz, z \mapsto tz + 1/j\}$. $\Phi_t := (\Phi_t^{(j)})_{j=1}^{\infty}$

Main Theorem B

The family $\{\Phi_t\}_{t\in U}$ of parameterized systems is a TNCIFS but Φ_t does not satisfy the separation condition for any $t \in U$.

Separation condition: Let $X \subset \mathbb{R}^m$ be a compact subset such that $\phi_{i,t}^{(j)}(X) \subset X$ for any $i \in \{1, 2\}$ and any $j \in \mathbb{N}$. Then we say that Φ_t satisfies the separation condition if for all $j \in \mathbb{N}$,

$$\phi_{1,t}^{(j)}(X) \cap \phi_{2,t}^{(j)}(X) = \emptyset.$$

Hausdorff dimension of the limit set of Φ_t

Let J_t be the limit set of Φ_t. For any $s \in [0, \infty)$,

$$\underline{P}_t(s) = \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in I^n} (||D\phi_{\omega,t}||_X)^s$$
$$= \liminf_{n\to\infty} \frac{1}{n} \log(2^n |t|^{ns}) = \log 2 + s \log |t|.$$

Hence for each $t \in U$, $\underline{P}_t(s)$ has the zero

$$s(t) = \frac{\log 2}{-\log |t|} (< 2).$$

- Since $\{\Phi_t\}_{t\in U}$ is a TNCIFS, we have

$$\dim_H(J_t) = s(t)$$

for a.e. $t \in U$ by Main Theorem A.

Outline of the proof of Main Theorem B

- We prove a slight variation of a technical lemma for the transversality condition given by Solomyak, 1998.
- Let $G \subset U$ be a compact subset.

$$\mathcal{L}_2(\{t \in G \ : \ |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq r\}) \leq (Kn^2)r^2,$$

$$\frac{1}{n}\log(Kn^2) = \frac{1}{n}\log K + \frac{2}{n}\log n \to 0$$

as $n \to \infty$.

- A power series of the form $1 + \sum_{j=1}^{\infty} a_j z^j$, with $a_j \in [-1, 1]$, cannot have a non-real double zero of modulus less than $2 \times 5^{-5/8} \approx 0.73143 (> 1/\sqrt{2})$ (Beaucoup, Borwein, Boyd and Pinner, 1998, Solomyak and Xu,2003).

Reference

- Yuto Nakajima, Transversal family of non-autonomous conformal iterated function systems, J. Fractal Geom. 2024, published online first DOI 10.4171/JFG/144. (arXiv:2308.13213)

Thank you!