Is the penalty function of the hard square shift stable?

Chihiro Oguri

Ochanomizu University

May 31, 2024

Joint work with Mao Shinoda
(Ochanomizu University)

Contents

Introduction

Ergodic Optimization

$T : X \to X$ continuous map on a compact metric space X.
$f : X \to \mathbb{R}$: continuous function (potential)

Find a T-inv measure μ which maxmize

$$\int f d\mu.$$

We call it a maximizing measure of f.

Remark

In this setting there exists at least one maximizing measure for a continuous function.

Introduction

- We consider symbolic dynamics, especially a fundamental class called SFTs and the potential that penalizes local configurations containing forbidden words for the given SFT.
- For the penalty potential, the maximizing measures are supported on the SFT.
- When the potential is perturbed, there is a difference between one-dimensional and two-dimensional.

The stability of penalty functions

	one-dimensional	two-dimensional
Gonschorowski et.al.	$^{\forall}\mathrm{SFT}$ The stability of the penalty function :○	$^{\exists}\mathrm{SFT}$ The stability of the penalty function :$\times$
		$^{\exists}\mathrm{SFT}$ is the Robinson tiling. $\rightarrow$ topological entropy is zero.

Question

Does the penalty function of the subshift of finite type (topological entropy is not zero) have the stability?

Setting of one-dimensional

$\mathcal{A}$:finite set (alphabet) $\mathcal{A}^{\mathbb{Z}}$:product space ($x \in \mathcal{A}^{\mathbb{Z}}$ $x = \{x_i\}_{i \in \mathbb{Z}}$)

- $\mathcal{A}^{\mathbb{Z}}$ is a compact metric space.

$$d(x, y) = \frac{1}{2^i} \ (i = \inf\{|i| \in \mathbb{Z}_{\geq 0} | x_i \neq y_i\})$$

- Define the shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ as

$$\sigma(\{x_i\}) = \{x_{i+1}\}.$$

- $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the full-shift.
- $w = w_0 w_1 \cdots w_{n-1} \in \mathcal{A}^n$,we call w a word of length n.

Setting of one-dimensional

definition 1 (Subshift)

$X \subset \mathcal{A}^{\mathbb{Z}}, \sigma$:*the shift map of* $\mathcal{A}^{\mathbb{Z}}$

(X, σ_X) *is called subshift if* X *satisfis the following two conditions:*

- X *is a closed set.*
- $\sigma^{-1}X = X$

definition 2 (Cylinder set)

$w = w_0 w_1 \cdots w_{n-1} \in \mathcal{A}^n$

$$[w] = \left\{ x \in \mathcal{A}^{\mathbb{Z}} | x_i = w_i \text{ for } i = 0, 1, \cdots, n-1 \right\}$$

- We call $[w]$ the cylinder set.

Setting of one-dimensional

definition 3 (subshift of finite type)

A subshift X is the subshift of finite type

$\overset{\text{def}}{\Longleftrightarrow}$

$^{\exists}$*finite set* $F \subset \bigcup_{n\geq 1} \mathcal{A}^n$ *s.t.*

$X = \{x \in \mathcal{A}^{\mathbb{Z}} : \sigma^n(x) \notin \bigcup_{w\in F}[w] \ ^{\forall}n \in \mathbb{Z}\}$

- We call F the set of forbidden words.
- $||f||_{Lip} := ||f||_{\infty} + c(f)$
  - $||f||_{\infty} = \sup|f|$
  - $c(f)$:The smallest Lipschitz constant of f

The stability of one-dimensional

Theorem 1 ([Gonschorowski et.al. 2021])

- F : *the set of forbidden words of length 2*
- $X = \mathrm{SFT}(F) \subset \mathcal{A}^{\mathbb{Z}}$:*one-dimensional aperiodic irreducible subshift of finite type.*
- f: *the penalty function (Lipschitz) defined by :*

$$f(x) = \begin{cases} -1 & \text{if } x_0x_1 \in F \\ 0 & \text{otherwise} \end{cases} .$$

- $\exists\varepsilon > 0 \ s.t. \ \forall g \in \mathrm{Lip}(\mathcal{A}^{\mathbb{Z}})$ *with* $||f - g||_{Lip} < \varepsilon$,

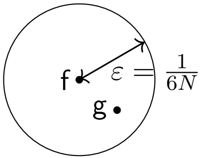
  Every $g-$*maximizing measure is supported on* X.

Ideas of the proof of Theorem 1

- X:Aperiodic irreducible subshift of finite type

$\rightarrow$ $^{\exists}N \geq 0\ s.t.\ ^{\forall}i, j \in \mathcal{A},\ ^{\exists}w$ word with length N s.t. there is no forbidden word in iwj.

- Let $\varepsilon = \dfrac{1}{6N}$



- Show that every maximizing measure of g is supported on X.

Ideas of the proof of Theorem 1

$\star$Goal of the proof

- μ:invariant measure supported on X^c
- g:Lipschitz function ($||f-g||_{Lip}<\varepsilon$)

$\rightarrow$ Show that there exists an invariant probability measure ν supported on X such that

$$\int g d\mu < \int g d\nu.$$

$I=\{x\in X|f(x)=0\}$

(i) $\mu(I^c)\geq\frac{1}{2N}$

(ii) $\mu(I^c)<\frac{1}{2N}$

- Use the slicing and coupling technique.

Sketch of the coupling and slicing

- μ: ergodic
- By the Birkoff ergodic theorem we have

$$\lim_{n\to+\infty}\frac{1}{n}\#\{i\in\{0,1,\cdots,n-1\}\mid \sigma^i x\in I^c\}$$

$$=\mu(I^c)<\frac{1}{2N}\ \mu\text{-a.e.}x.$$

- Take x satisfing above.

Sketch of the coupling and slicing

- Marking $i \in \mathbb{N}$ s.t. $\sigma^i x \in I^c$, we make "bad blocks".
- we replace the words in bad blocks and check this implies

$$\int g d\mu < \int g d\nu$$

for some σ-invariant measure ν supported on X.

Setting of two-dimensional

$\mathcal{A}$:finite set

- $\mathcal{A}^{\mathbb{Z}^2} \overset{\sigma}{\curvearrowleft} \mathbb{Z}^2$ where $\sigma^{(n,m)}(\{x_{(i,j)}\}_{(i,j)\in\mathbb{Z}^2}) = x_{(i+n,j+m)}$
- For each $n \geq 1$

$$\Lambda_n := [-n, n] \times [-n, n] \cap \mathbb{Z}^2$$

- For $x \in A^{\mathbb{Z}^2}$ x_{Λ_n}denotes the restriction of x on Λ_n

definition 4

X *is the subshift fo finite type if* $^{\exists}n \geq 1,\ ^{\exists}F \subset \mathcal{A}^{\Lambda_n}$ *s.t.*

$$X = \{x \in \mathcal{A}^{\mathbb{Z}^2} | (\sigma^{(u_1,u_2)}(x))_{\Lambda_n} \notin F,\ ^{\forall}(u_1, u_2) \in \mathbb{Z}^2\}$$

The stability of two-dimensional

Theorem 2 ([Gonschorowski et.al. 2021])

- *There exists a shift of finite type* $X = \mathrm{SFT}(F)$
- F*:the set of 2 × 2 forbidden blocks*
- f*:the penalty function defined by :*

$$f(x) = \begin{cases} -1 & \text{if } \begin{smallmatrix} x_{01} & x_{11} \\ x_{00} & x_{10} \end{smallmatrix} \in F \\ 0 & \text{otherwise} \end{cases} .$$

- $^{\forall}\varepsilon > 0,\ ^{\exists}g \in \mathrm{Lip}(\mathcal{A}^{\mathbb{Z}^2})$ *with* $||f - g||_{Lip} < \varepsilon$ *and* $^{\exists}g-$*maxmizing measure* μ_g *is supported on* X^c.

- The SFT in the previous Theorem is the Robinson tiling.
  - The Robinson tiling has no periodic point.
  - The topological entropy of the Robinson tiling is zero.
- Dynamical properties of the Robinson tiling are much different from one-dimensional SFT.

→ We pay attention to the two-dimensional SFT with

- "many" periodic points
- positive topological entropy.

- It is natural to ask whether the stability holds for a SFT where positive topological entropy.

The hard square shift

$\mathcal{A} = \{0, 1\}$

$\mathcal{A}^{\mathbb{Z}^2} = \left\{ \left(x_{(n,m)}\right)_{(n,m)\in\mathbb{Z}^2} \mid x_{(n,m)} \in \mathcal{A},\ (n,m) \in \mathbb{Z}^2 \right\}$

$X =$SFT(F) where $F = \left\{ \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\ \begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\ \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\ \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\ \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\ \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\ \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\ \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\ \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix} \right\}$

(F is the set of forbidden blocks)

Define the penalty function

$$f(x) = \begin{cases} -1 & \begin{smallmatrix} x_{10} & x_{11} \\ x_{00} & x_{01} \end{smallmatrix} \in F \\ 0 & \text{otherwise} \end{cases}$$

Problem

$^{\exists}\varepsilon > 0$ s.t. $^{\forall}g \in \mathrm{Lip}(\mathcal{A}^{\mathbb{Z}^2})$ with $||f - g||_{\mathrm{Lip}} < \varepsilon$

Every $g-$maximizing measure is supported on X.

References

Juliano S. Gonschorowski, Anthony Quas, & Jason Siefken. (2021). Support Stability of Maximizing Measures for Shifts of Finite Type. Ergodic Theory and Dynamical Systems. 41(3). pp. 869 - 880

BRIAN MARCUS & RONNIE PAVLOV. (2013). Approximating entropy for a class of $\mathbb{Z}^2$ Markov random fields and pressure for a class of functions on $\mathbb{Z}^2$ shifts of finite type. Ergodic Theory and Dynamical Systems. 33(1). pp. 186 - 220