Relative Dynamics and Stability of Point Vortices

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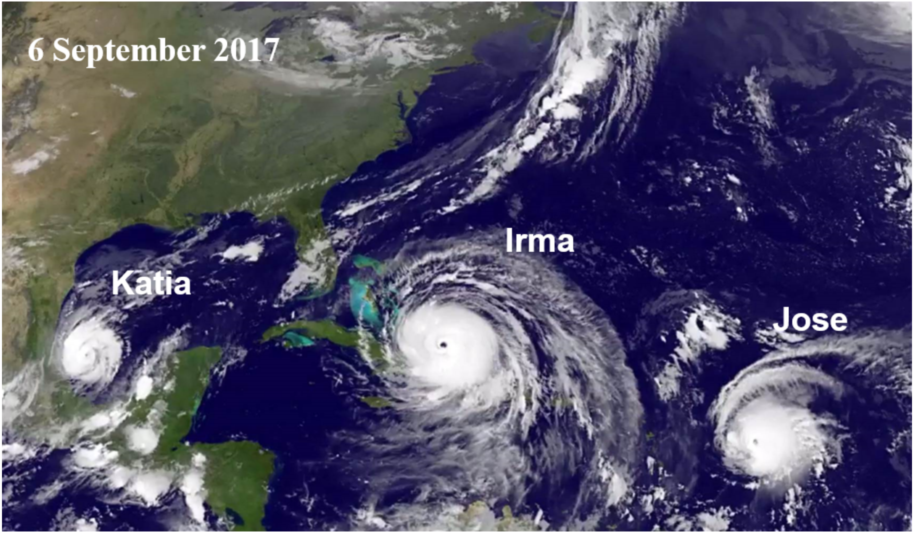
Dynamics of Hurricanes?



Source: U. Washington News & NOAA

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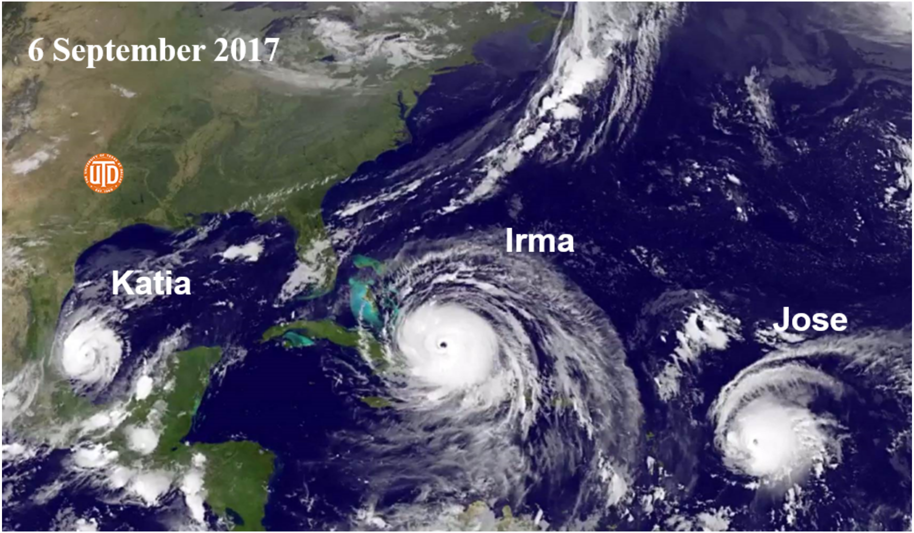

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Point Vortex on $\mathbb{R}^2$

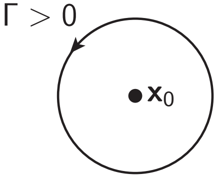
$$\text{Point vortex with circulation } \Gamma \text{ at } \mathbf{x}_0 = (x_0, y_0)$$

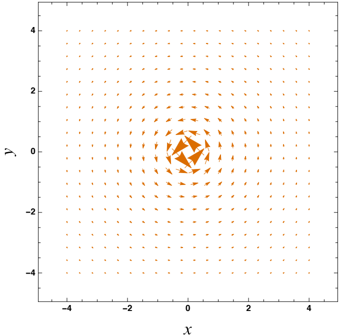
$$\updownarrow$$

$$\text{Vorticity } \xi(\mathbf{x}) = \nabla \times \mathbf{u}(\mathbf{x}) = \Gamma\, \delta(\mathbf{x} - \mathbf{x}_0)$$

With $\nabla \cdot \mathbf{u} = 0$,

$$\mathbf{u}(\mathbf{x}) = \frac{\Gamma}{2\pi \left\|\mathbf{x} - \mathbf{x}_0\right\|^2}(-(y - y_0), x - x_0)$$

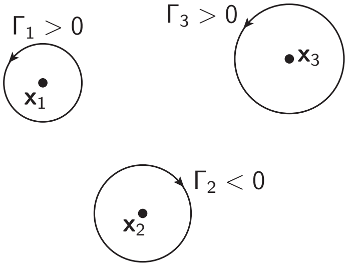




Dynamics of N Point Vortices on $\mathbb{R}^2$

Each point vortex j located at $\mathbf{x}_j \in \mathbb{R}^2$ is convected by the net velocity of the other vortices:

$$\dot{\mathbf{x}}_j(t) = \sum_{\substack{1 \le k \le N \\ k \neq j}} \mathbf{u}_k(\mathbf{x}_j(t)),$$



which gives, writing $\mathbf{x}_j = (x_j, y_j)$,

$$\dot{x}_j = -\frac{1}{2\pi} \sum_{\substack{1 \le k \le N \\ k \neq j}} \Gamma_k \frac{y_j - y_k}{\|\mathbf{x}_j - \mathbf{x}_k\|^2}, \qquad \dot{y}_j = \frac{1}{2\pi} \sum_{\substack{1 \le k \le N \\ k \neq j}} \Gamma_k \frac{x_j - x_k}{\|\mathbf{x}_j - \mathbf{x}_k\|^2}.$$

Dynamics of N Point Vortices on $\mathbb{R}^2$

Or, by setting $q_j := x_j + \mathrm{i} y_j \in \mathbb{C}$,

$$\dot{q}_j = \frac{\mathrm{i}}{2\pi} \sum_{\substack{1 \le k \le N \\ k \ne j}} \Gamma_k \frac{q_j - q_k}{|q_j - q_k|^2}.$$

Also a Hamiltonian system: Writing

$$r_j := \sqrt{|\Gamma_j|}\, x_j, \qquad p_j := \operatorname{sgn}(\Gamma_j) \sqrt{|\Gamma_j|}\, y_j,$$

we have

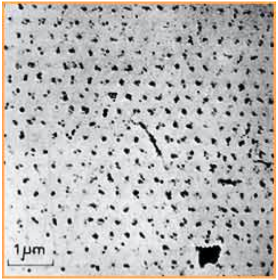
$$\dot{r}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial r_j},$$

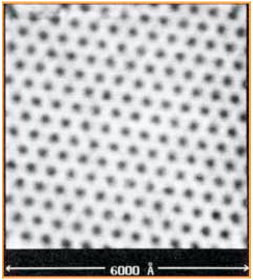
where

$$H(r, p) := -\frac{1}{4\pi} \sum_{1 \le i < j \le N} \Gamma_i \Gamma_j \ln \|\mathbf{x}_i - \mathbf{x}_j\|^2 .$$

Applications of Point Vortex Dynamics

- Fluid dynamics
- Superfluidity and superconductivity



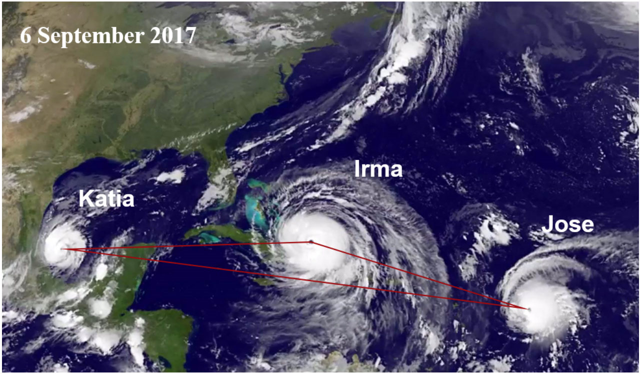


Abrikosov vortices in Type II superconductor from Essmann and Trauble (1967)

Relative/Shape Dynamics of Hurricanes?

Goal

Dynamics of "shape" of N vortices (regardless of its position and orientation)?

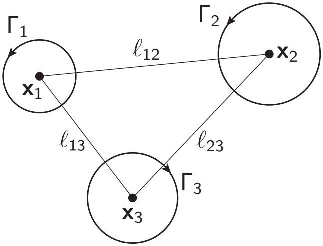


E.g., if $N = 3$, "shape" of 3 vortices = triangle formed by them

Equations of Relative Motion?

Inter-vortex distance:

$$\ell_{jk} := \|\mathbf{x}_j - \mathbf{x}_k\|$$



Equations of Relative Motion (Newton, Aref,...):

$$\frac{d}{dt}\ell_{jk}^2 = \frac{2}{\pi} \sum_{\substack{1 \le l \le N \\ l \ne j, l \ne k}} \Gamma_l A_{jkl} \left(\frac{1}{\ell_{kl}^2} - \frac{1}{\ell_{jl}^2} \right),$$

$$\frac{d}{dt} A_{jkl} = ???$$

where A_{jkl} := signed area of vortex triangle jkl.

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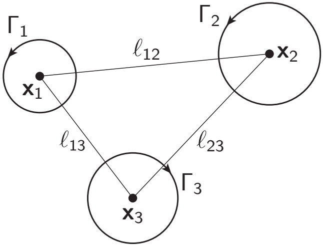

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Question

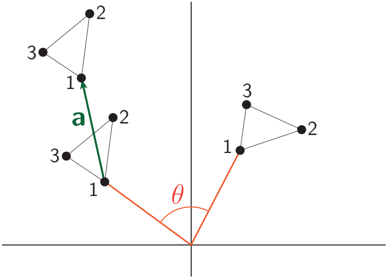
Hamiltonian formulation of relative dynamics?

SE(2)-**Action on the Plane** $\mathbb{R}^2$

- Symmetry group

$$\begin{aligned} \mathrm{SE}(2) &:= \mathrm{SO}(2) \ltimes \mathbb{R}^2 \\ &= \text{All rotations of } \mathbb{R}^2 \text{ and translations of } \mathbb{R}^2 \text{ combined} \end{aligned}$$

- SE(2)-action on $\mathbb{R}^2$:



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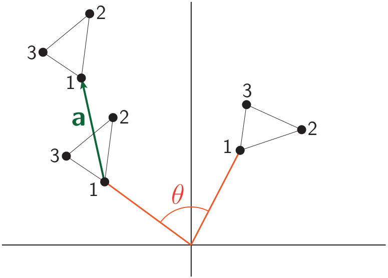


$$\textbf{Relative dynamics} = \frac{\textbf{Original dynamics}}{\textbf{SE(2)}}$$

Reduction by $\mathbb{R}^2$: 1st Stage of $\mathsf{SE}(2)$-Reduction

- Translational action by $\mathbb{R}^2 \cong \mathbb{C}$:

$$\mathbb{C} \times \mathbb{C}^N \to \mathbb{C}^N; \qquad (a, \mathbf{q} := (q_1, \dots, q_N)) \mapsto (q_1 + a, \dots, q_N + a)$$

- Momentum map (conserved quantity):

$$\mathbf{I}(\mathbf{q}) := -\mathrm{i} \sum_{j=1}^{N} \Gamma_j q_j \quad (\text{"linear impulse"})$$

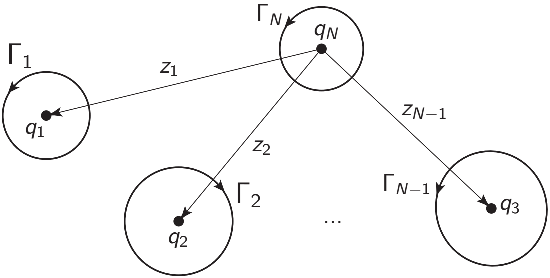
$\mathbb{R}^2$-Reduced Space

For $\Gamma \neq 0$, $\mathbb{R}^2$-Reduced Space:

$$\begin{aligned} Z &:= \mathbf{I}^{-1}(0) \cong \mathbb{C}^{N-1} \\ &= \{(z_1, \dots, z_{N-1})\}, \end{aligned}$$

where z_j's are relative coordinates w.r.t. last vortex:

$$(z_1, \dots, z_{N-1}) := (q_1 - q_N, \dots, q_{N-1} - q_N).$$



Reduction by SO(2): 2nd Stage of SE(2)-Reduction

- $\mathsf{SO}(2) \cong \mathbb{S}^1$-action on $\mathbb{C}^{N-1}$:

$$\mathbb{S}^1 \times \mathbb{C}^{N-1} \to \mathbb{C}^{N-1}; \ \left(e^{\mathrm{i}\theta}, z = (z_1, \dots, z_{N-1})\right) \mapsto \left(e^{\mathrm{i}\theta} z_1, \dots, e^{\mathrm{i}\theta} z_{N-1}\right)$$

- Momentum map (conserved quantity):

$$K(z) = -\frac{1}{2} z^* \mathcal{K} z \quad (\text{“angular impulse”}),$$

where $\mathcal{K}$ is a non-singular matrix depending on $\{\Gamma_j\}_{j=1}^N$.

- Reduced space $K^{-1}(c_0)/\mathbb{S}^1$; this is where the **relative dynamics** is.

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Problem

$K^{-1}(c_0)/\mathbb{S}^1$ is a rather awkward space to work with.

Getting Around $K^{-1}(c_0)/\mathbb{S}^1$ via Symplectic Geometry

Recall:

$$Z = \{(z_1, \dots, z_{N-1})\} \cong (\mathbb{C}\backslash\{0\})^{N-1} \quad \text{and} \quad K(z) = -\frac{1}{2} z^* \mathcal{K} z.$$

Getting Around $K^{-1}(c_0)/\mathbb{S}^1$ via Symplectic Geometry

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Consider Lie group

$$\mathsf{U}(\mathcal{K}) = \left\{ U \in \mathbb{C}^{(N-1)\times(N-1)} \mid U^* \mathcal{K} U = \mathcal{K} \right\}$$

and its Lie algebra

$$\begin{aligned} \mathfrak{u}(\mathcal{K}) &= \left\{ \tilde{\mu} \in \mathbb{C}^{(N-1)\times(N-1)} \mid \tilde{\mu}^* \mathcal{K} + \mathcal{K}\tilde{\mu} = 0 \right\} \\ &\cong \left\{ \mathrm{i}\mu \in \mathbb{C}^{(N-1)\times(N-1)} \mid \mu^* = \mu \right\} \cong \mathfrak{u}(N) \end{aligned}$$

via $\mathrm{i}\mu := \mathcal{K}\tilde{\mu}$, and consider

$$\mathbf{J} \colon Z \to \mathfrak{u}(\mathcal{K})^* \cong \mathfrak{u}(\mathcal{K}); \qquad z \mapsto \mathrm{i} z z^*,$$

Getting Around $K^{-1}(c_0)/\mathbb{S}^1$ via Symplectic Geometry

$$
\begin{array}{ccc}
z & & \\
\downarrow & \searrow & \\
[z] & \longmapsto & \mathrm{i}zz^* =: \mathrm{i}\mu
\end{array}
\qquad\qquad
\begin{array}{ccc}
K^{-1}(c_0) & & \\
\downarrow \pi_{c_0} & \searrow^{\mathbf{J}|_{K^{-1}(c_0)}} & \\
K^{-1}(c_0)/\mathbb{S}^1 & \xrightarrow[\bar{\mathbf{J}}]{} & \mathcal{O}_{\mu_0} \subset \mathfrak{u}(\mathcal{K})^*
\end{array}
$$

Then

$$K^{-1}(c_0)/\mathbb{S}^1 \cong \text{coadjoint orbit } \mathcal{O}_{\mu_0} \text{ in } \mathfrak{u}(\mathcal{K})^*,$$

where

$$\mathcal{O}_{\mu_0} = \left\{ U\mu_0 U^* \in \mathbb{C}^{(N-1)\times(N-1)} \mid U \in \mathsf{U}(\mathcal{K}) \right\} \subset \underbrace{\mathfrak{u}(\mathcal{K})^*}_{\textbf{vector space!}}$$

Hamiltonian Formulation of Relative Dynamics

Theorem

1. The **relative dynamics** of N point vortices with non-vanishing angular impulse is governed by a Lie–Poisson equation in $\mathfrak{u}(\mathcal{K})^* \cong \mathfrak{u}(N)$:

$$\dot{\mu} = -\operatorname{ad}^*_{Dh(\mu)} \mu = -\mu Dh(\mu)\mathcal{K}^{-1} + \mathcal{K}^{-1} Dh(\mu)\mu,$$

   where $\mu := \mathbf{J}(z) = \mathrm{i} z z^*$ and h is the Hamiltonian, i.e., $H = h \circ \mathbf{J}$.

2. $C_j(\mu) := \operatorname{tr}((\mathrm{i}\,\mathcal{K}\mu)^j)$ is a **Casimir (conserved quantity)** for any $j \in \{1, \dots, N-1\}$.

Remark

Lie–Poisson equations are a special class of Hamiltonian systems defined on the dual of a Lie algebra.

Example: $N = 3$

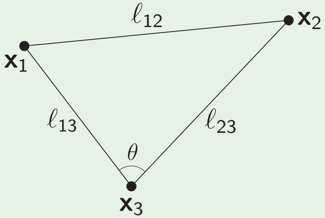
Example: Relative Dynamics for $N = 3$

$$\dot{\mu} = -\operatorname{ad}^*_{Dh(\mu)} \mu \quad \text{with} \quad \mu = \mathrm{i} \begin{bmatrix} \mu_1 & \mu_3 + \mathrm{i}\,\mu_4 \\ \mu_3 - \mathrm{i}\,\mu_4 & \mu_2 \end{bmatrix} \in \mathfrak{u}(\mathcal{K})^* \cong \mathfrak{u}(2)$$

where

$$\mu_1 = \ell_{13}^2 \qquad \mu_2 = \ell_{23}^2$$

$$\mu_3 + \mathrm{i}\,\mu_4 = \ell_{13}\ell_{23} e^{\mathrm{i}\theta_3}$$



are the shape variables.

Application of Relative Dynamics

Next Goal

Apply the above formulation to stability of **relative equilibria**.

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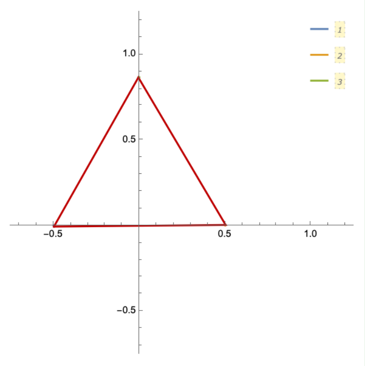
A **relative equilibrium** is a solution $t \mapsto \{\mathbf{x}_j(t)\}_{j=1}^N$ of the original N-vortex system where each $\{\mathbf{x}_j(t)\}_{j=1}^N$ is obtained by a rigid (Euclidean) transformation of the initial point $\{\mathbf{x}_j(0)\}^N$.

Example of Relative Equilibrium

Example ($N = 3$; Equilateral Triangle)

$(\Gamma_1, \Gamma_2, \Gamma_3)$ being at the vertices of an equilateral triangle gives a relative equilibrium.

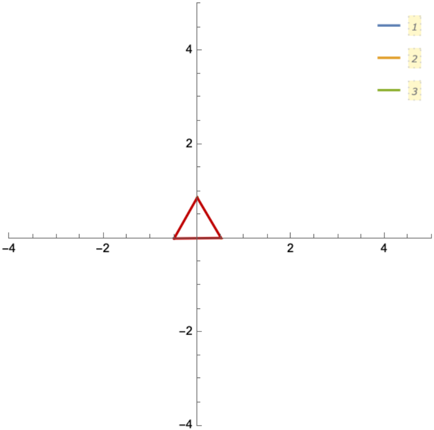
$(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 2, 3)$



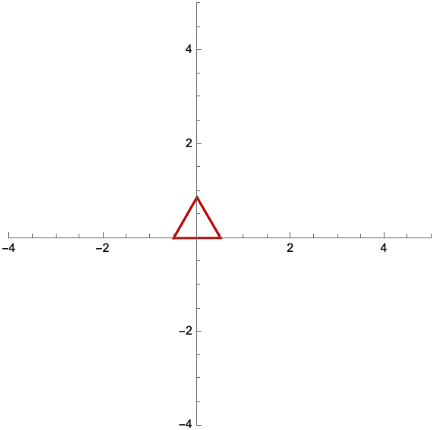
Stability of Relative Equilibria

The stability of equilateral triangle relative equilibria depends on the circulations:

$(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 2, 3)$



$(\Gamma_1, \Gamma_2, \Gamma_3) = (1, -2, 3)$



Relative Equilibria is a Fixed Point in Relative Dynamics

Relative equilibrium of $\dot{q}_j = \dfrac{\mathrm{i}}{2\pi} \sum_{\substack{1 \le k \le N \\ k \ne j}} \Gamma_k \dfrac{q_j - q_k}{|q_j - q_k|^2}$

$$\Updownarrow$$

Fixed point of $\dot{\mu} = -\operatorname{ad}^*_{Dh(\mu)} \mu$

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$$\Updownarrow$$

$$\textbf{Fixed point} \text{ of } \dot{\mu} = -\operatorname{ad}^*_{Dh(\mu)} \mu$$

Main Idea

Analyze the stability of the former by doing it for the latter.

Main Drawback of Lie–Poisson Relative Dynamics

No free lunch!

- **Advantage**: Our relative dynamics is defined on a vector space $\mathfrak{u}(N)$ (as opposed to a complicated manifold)
- **Disadvantage**: The matrix μ becomes huge as N increases:

$$\mu = \mathrm{i}\begin{bmatrix} \mu_1 & \mu_{12} & \cdots & \mu_{1,N-1} \\ \mu_{12}^* & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mu_{N-2,N-1} \\ \mu_{1,N-1}^* & \cdots\ \mu_{N-2,N-1}^* & & \mu_{N-1} \end{bmatrix} \in \mathfrak{u}(N) \cong \mathbb{R}^{(N-1)^2},$$

i.e., μ has many redundant variables—the price we pay for formulating the dynamics in a vector space.

Rank-1 Constraint in Lie–Poisson Relative Dynamics

Recall that μ is defined as

$$\mu = \mathrm{i} \underbrace{zz^*}_{\text{Hermitian}} \text{ with } z := \begin{bmatrix} q_1 - q_N \\ \dots \\ q_{N-1} - q_N \end{bmatrix} \neq 0 \text{ ,i.e., } \operatorname{rank} \mu = 1.$$

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Lemma

Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ be a Hermitian matrix with no vanishing elements with $n \geq 2$. Then rank $A = 1$ *iff the determinants of all the 2×2 submatrices shown below vanish.*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ a_{41} & a_{42} & a_{43} & \ddots & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & \cdots & a_{nn} \end{bmatrix}$$

Constraints in Relative Dynamics

Proposition

Let

$$\mathring{\mathfrak{u}}(N) := \{\mu \in \mathfrak{u}(N) \mid \textit{all entries of } \mu \textit{ are non-zero}\}$$

and set

$$R \colon \mathring{\mathfrak{u}}(N) \to \mathbb{R}^{(N-2)^2};$$
$$\mu \mapsto \textit{all real and imaginary parts of the above determinants.}$$

Then the relative dynamics $t \mapsto \mu(t)$ is constrained in the submanifold

$$\{\mu \in \mathring{\mathfrak{u}}(N) \mid \operatorname{rank}\mu = 1\} = R^{-1}(0).$$

of dimension $2N-3$.

Summary of Lie–Poisson Relative Dynamics

- **Dynamics**: Lie–Poisson equation (special class of Hamiltonian system) in $\mathfrak{u}(N)$:

$$\dot{\mu} = -\operatorname{ad}^*_{Dh(\mu)} \mu$$

- **Invariants**: Casimirs

$$C_j(\mu) := \operatorname{tr}((\mathrm{i}\,\mathcal{K}\mu)^j) \quad \forall j \in \{1, \dots, N-1\}$$

- **Constraints**: The dynamics is constrained to the zero level set

$$R^{-1}(0) = \{\mu \in \mathring{\mathfrak{u}}(N) \mid \operatorname{rank} \mu = 1\}$$

of function R taking values in $\mathbb{R}^{(N-2)^2}$.

Stability of Relative Equilibria

Theorem (Stability Condition for Relative Equilibria)

Let $\mu_0 \in R^{-1}(0)$ be a fixed point of the Lie–Poisson dynamics, and $\{C_j\}_{j=1}^{K}$ be a subset of the Casimirs $\{C_j\}_{j=1}^{N-1}$ such that $\{C_j\}_{j=1}^{K} \cup \{R\}$ are independent at μ_0. Suppose that there exist constants $a_0 \in \mathbb{R}\backslash\{0\}$, $\{a_i \in \mathbb{R}\}_{i=1}^{K}$ and $\{b_i \in \mathbb{R}\}_{i=1}^{(N-2)^2}$ such that

$$f(\mu) := a_0 h(\mu) + \sum_{i=1}^{K} a_i C_i(\mu) + \sum_{i=1}^{(N-2)^2} b_i R_i(\mu)$$

satisfies the following:

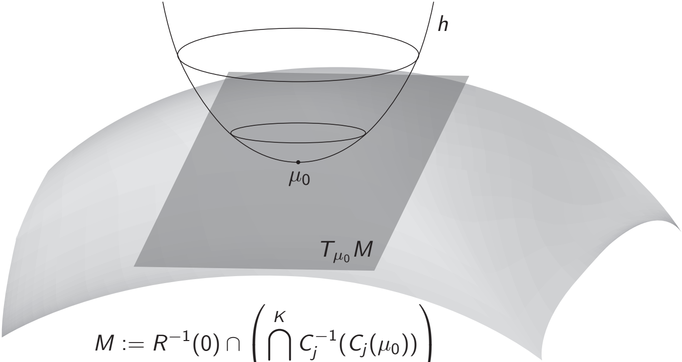
1. *$Df(\mu_0) = 0$; and*
2. *the Hessian $D^2 f(\mu_0)$ is positive definite on the tangent space at μ_0 of the level set*

$$M := R^{-1}(0) \cap \left(\bigcap_{j=1}^{K} C_j^{-1}(C_j(\mu_0)) \right).$$

Then μ_0 is Lyapunov stable.

Stability of Relative Equilibria

Intuitive idea behind the theorem:



$$M := R^{-1}(0) \cap \left(\bigcap_{j=1}^{K} C_j^{-1}(C_j(\mu_0)) \right)$$

The Hamiltonian h (which is an invariant of dynamics) takes a local minimum at fixed point μ_0 on M.

Example: Stability of Equilateral Triangle

$$\mu = \mathrm{i}\begin{bmatrix} \mu_1 & \mu_3 + \mathrm{i}\,\mu_4 \\ \mu_3 - \mathrm{i}\,\mu_4 & \mu_2 \end{bmatrix} = (\mu_1, \mu_2, \mu_3, \mu_4)$$

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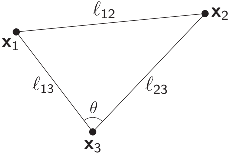

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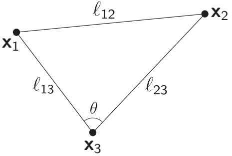
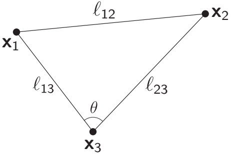


- Hamiltonian
  $h(\mu) = -\frac{1}{4\pi}\left(\Gamma_1\Gamma_3 \ln \mu_1 + \Gamma_2\Gamma_3 \ln \mu_2 + \Gamma_1\Gamma_2 \ln(\mu_1 + \mu_2 - 2\mu_3)\right)$
- Casimir $C_1(\mu) = \frac{\Gamma_2(\Gamma_1+\Gamma_3)\mu_1 + \Gamma_1(\Gamma_2+\Gamma_3)\mu_2 - 2\Gamma_1\Gamma_2\mu_3}{\Gamma_1+\Gamma_2+\Gamma_3}$
- Constraint $R(\mu) = \det \mu = \mu_1\mu_2 - \mu_3^2 - \mu_4^2 = 0$
- Fixed point $\mu_0 := (1, 1, 1/2, -\sqrt{3}/2)$

Example: Stability of Equilateral Triangle

$$\mu = \mathrm{i}\begin{bmatrix} \mu_1 & \mu_3 + \mathrm{i}\,\mu_4 \\ \mu_3 - \mathrm{i}\,\mu_4 & \mu_2 \end{bmatrix} = (\mu_1, \mu_2, \mu_3, \mu_4)$$

$$\mu_1 = \ell_{13}^2 \qquad \mu_2 = \ell_{23}^2$$
$$\mu_3 + \mathrm{i}\,\mu_4 = \ell_{13}\ell_{23}e^{\mathrm{i}\theta_3}$$



- Hamiltonian
  $h(\mu) = -\frac{1}{4\pi}\left(\Gamma_1\Gamma_3 \ln \mu_1 + \Gamma_2\Gamma_3 \ln \mu_2 + \Gamma_1\Gamma_2 \ln(\mu_1 + \mu_2 - 2\mu_3)\right)$
- Casimir $C_1(\mu) = \frac{\Gamma_2(\Gamma_1+\Gamma_3)\mu_1+\Gamma_1(\Gamma_2+\Gamma_3)\mu_2-2\Gamma_1\Gamma_2\mu_3}{\Gamma_1+\Gamma_2+\Gamma_3}$
- Constraint $R(\mu) = \det\mu = \mu_1\mu_2 - \mu_3^2 - \mu_4^2 = 0$
- Fixed point $\mu_0 := (1, 1, 1/2, -\sqrt{3}/2)$

If $\Gamma_1\Gamma_2 + \Gamma_1\Gamma_3 + \Gamma_2\Gamma_3 > 0$, one can find a_0, a_1, b_1 such that

$$f(\mu) := a_0 h(\mu) + a_1 C_1(\mu) + b_1 R(\mu)$$

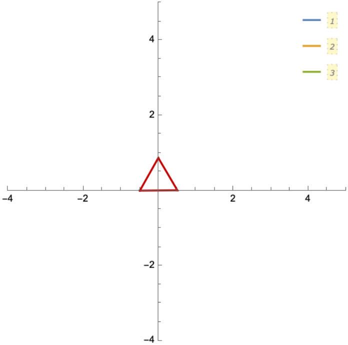
satisfies (i) $Df(\mu_0) = 0$ and (ii) $D^2 f(\mu_0) > 0$ on the tangent space at μ_0 of $R^{-1}(0) \cap C_1^{-1}(C_1(\mu_0))$.

Example: Stability of Equilateral Triangle

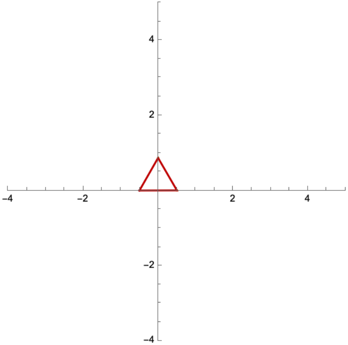
Proposition (Reproducing Synge (1949) and Aref (1979))

An equilateral triangle relative equilibrium is stable if $\Gamma_1\Gamma_2 + \Gamma_1\Gamma_3 + \Gamma_2\Gamma_3 > 0$ *and is unstable if* $\Gamma_1\Gamma_2 + \Gamma_1\Gamma_3 + \Gamma_2\Gamma_3 < 0$.

$$(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 2, 3)$$
$$\implies \Gamma_1\Gamma_2 + \Gamma_1\Gamma_3 + \Gamma_2\Gamma_3 = 11 > 0$$

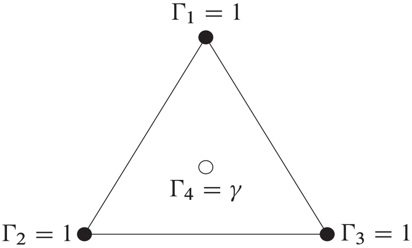


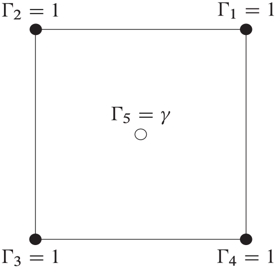
$$(\Gamma_1, \Gamma_2, \Gamma_3) = (1, -2, 3)$$
$$\implies \Gamma_1\Gamma_2 + \Gamma_1\Gamma_3 + \Gamma_2\Gamma_3 = -5 < 0$$



Examples: Equilateral Triangle with Center

Equilateral triangle/Square with center:





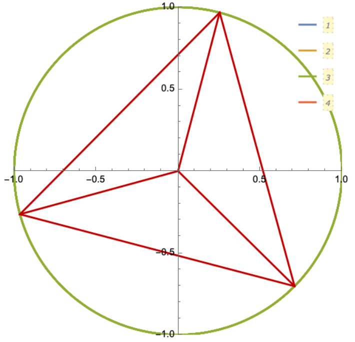
Relative equilibria $\forall \gamma \in \mathbb{R}$.

Example: Equilateral Triangle with Center

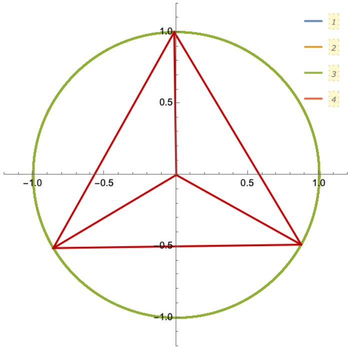
Proposition

Equilateral triangle with center is Lyapunov stable if $\gamma < -3$ *or* $0 < \gamma < 1$ *and linearly unstable if* $\gamma > 1$*.*

$\gamma = 1/2$



$\gamma = 3$

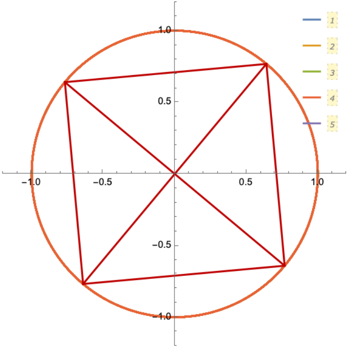


Example: Square with Center

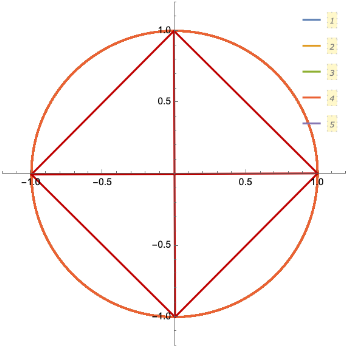
Proposition

Square with center is Lyapunov stable if $0 < \gamma < 9/4$ *and linearly unstable if* $\gamma < -1/2$ *or* $\gamma > 9/4$.

$\gamma = 2$



$\gamma = 3$



Summary

- Hamiltonian formulation of N-vortex relative dynamics
  - Dynamics in (the dual of) a Lie algebra, i.e., a vector space
  - with constraints and invariants
- Found a sufficient condition for stability of relative equilibria.
  - Can be used to derive stability condition in terms $\{\Gamma_i\}_{i=1}^N$
  - Used to find stability condition of following relative equlibria:
    - equilateral triangle with center
    - square with center