On the stochastic bifurcations regarding random iterations of rational maps

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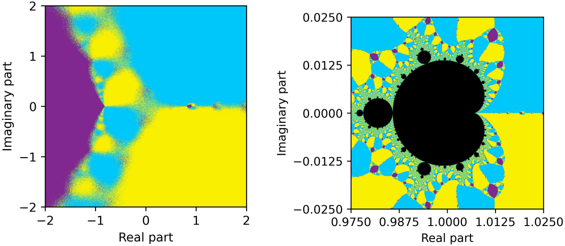


Introduction

I am interested in RANDOM dynamical systems. One of my motivations is the following theorem regarding Random Relaxed Newton's Methods:

Theorem 1 (Sumi '21)

If we insert randomness into **Newton's method** *in a suitable way, then we can find a root of a given function with probability one, for every but a finite number of initial points.*

Original Newton's method may fail if we choose a bad initial point. However, the theorem states that randomness benefits the algorithm.

Contents

I would like to talk about two topics:

Numerical observations of Random Relaxed Newton's Methods

Theoretical work on stochastic bifurcation of $\{z \mapsto z^2 + c\}_{c \in \mathbb{C}}$

Numerical observations of Random Relaxed Newton's Methods

Theoretical work on stochastic bifurcation of $\{z \mapsto z^2 + c\}_{c \in \mathbb{C}}$

Original Newton's method

For a given polynomial P, Newton's map is defined by

$$N_P(x) = x - \frac{P(x)}{P'(x)}.$$

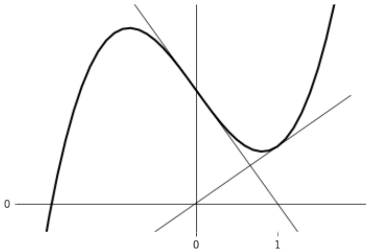
Lemma 2 (Dynamics of N_P on $\mathbb{C} \cup \{\infty\}$)

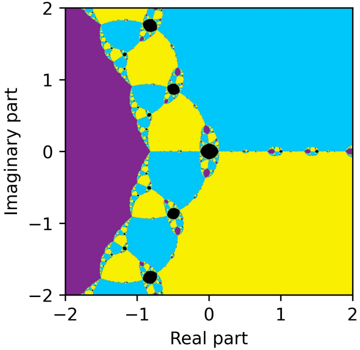
- *If $P(z^*) = 0$, then z^* is an attracting fixed point of N_P.*
- *If z^* is a simple root, then z^* is superattracting.*

This explains why Newton's method works well *locally*. But ...

Newton's method sometimes fails

If the target is $P(z) = z^3 - 2z + 2$, then N_P has a (super)attracting periodic cycle: $0 \mapsto 1 \mapsto 0$





P has three simple roots in $\mathbb{C}$. A non-black point belongs to a basin of attraction, and **the black part is the set of initial points which do not converge to any root**.

Every Newton-like method sometimes fails

This target $P(z) = z^3 - 2z + 2$ is not the only bad example, but there are many. More generally,

Theorem 3 (McMullen)

There is NO algebraic root-finding algorithms which almost always success.

We can overcome this by randomness.

Relaxed Newton's Methods

Definition 4 (Relaxed Newton's map)

For $\lambda \in \mathbb{C}$, define $N_{P,\lambda}(z) = z - \lambda \frac{P(z)}{P'(z)}$.
The original map is $N_P = N_{P,1}$.

Lemma 5

Every root z^* is a common attr fxd pt whenever $|1 - \lambda| < 1$.

We now randomly choose parameters $\lambda_1, \lambda_2, \ldots$ and define

$$z_{n+1} = N_{P,\lambda_{n+1}}(z_n) = z_n - \lambda_{n+1} \frac{P(z_n)}{P'(z_n)}.$$

Sample dynamics for $N_{P,\lambda}$ with $P(z) = z^3 - 2z + 2$

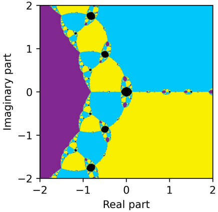


Figure: Original Newton's map has an attracting periodic point 0.

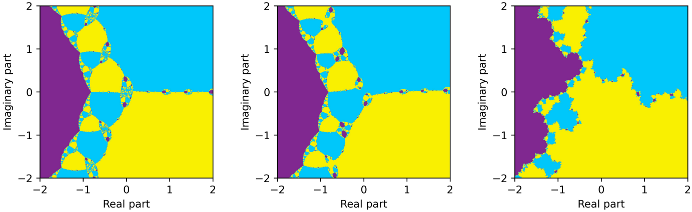


Figure: Three sample paths for Random Relaxed Newton's Methods (with more larger noise from left to right).

Random Relaxed Newton's Methods

Setting 6

Choose iid sequences λ_n following a probability measure μ on $\mathbb{C}$ which is absolutely continuous wrt 2-dim Lebesgue measure.

Typical examples of μ are the uniform distributions on disks $\bar{B}(1,\rho) = \{\lambda \in \mathbb{C} \colon |1-\lambda| \le \rho\}$: $\mu = \mathrm{Unif}(\bar{B}(1,\rho))$.

Theorem 7 (Sumi '21)

Suppose that $\bar{B}(1, 1/2) \Subset \operatorname{supp}\mu \Subset \bar{B}(1,1)$. *Then for every target polynomial* P *and for all initial points* z_0 *with finitely many exceptions, the random orbit* $N_{P,\lambda_n} \circ \cdots \circ N_{P,\lambda_2} \circ N_{P,\lambda_1}(z_0)$ *converges to some root of* P *with probability* 1.

In the following, we assume $\mu = \mathrm{Unif}(\bar{B}(1,\rho))$ for a while.

Averaged dynamics induced by $\mathrm{Unif}(\bar{B}(1,\rho))$

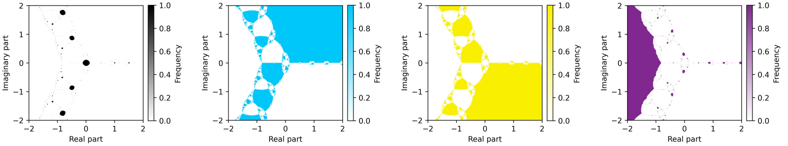


Figure: Deterministic case ($\rho = 0$). Four attracting basins.

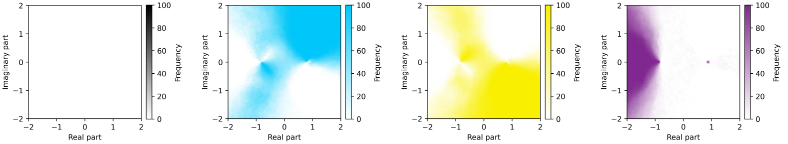


Figure: Random case ($\rho = 0.6$). For each initial point, 100 random orbits were computed and counted to see where they converge (or not).

The set of “bad initial points” collapses due to noise.

Notes from the dynamical viewpoint

Recall that $N_{P,\lambda}(z) = z - \lambda \frac{P(z)}{P'(z)}$.

Remark 8 (How does it work?)

Bounded and multiplicative noise keeps the roots of P as attracting fixed points. Conversely, sufficiently large noise affects undesirable attractors.

Question: What is the optimal value for the size of noise?
・If the size of noise is too large, then the speed of local convergence is slow.
・If the size of noise is too small, then the orbit cannot escape from bad attractors.

I would like to know the smallest noise size ρ such that the undesirable attractors disappear.

Numerical experiments with different noise size ρ

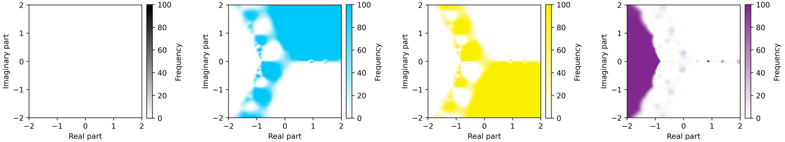


Figure: $\rho = 0.1$

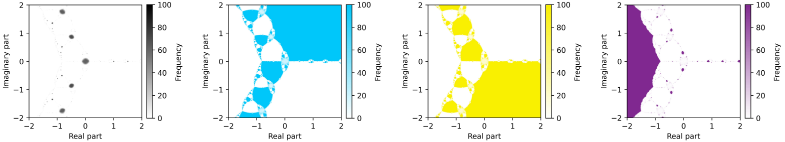


Figure: $\rho = 0.01$ (need high iterates)

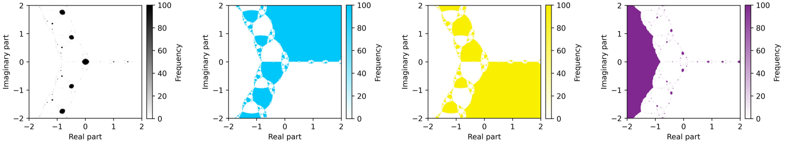


Figure: $\rho = 0.005$ (too small!)

Speed of convergence

Define random variables by

$$Z_n = N_{P,\bullet_n} \circ \cdots \circ N_{P,\bullet_2} \circ N_{P,\bullet_1}(z_0),$$
$$T = T(z_0, z^*, \epsilon) = \min\{n \in \mathbb{N} \colon |Z_n - z^*| < \epsilon\}$$

and calculate conditioned expectation $\mathbb{E}(T|T < \infty)$ for every root z^*.

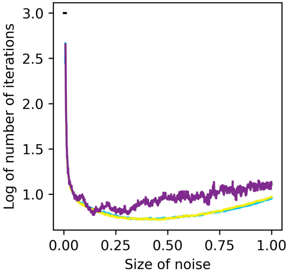


Figure: x-axis represents ρ, y-axis represents $\log_{10} \mathbb{E}(T|T < \infty)$ with initial point $z_0 = 0$ and $\epsilon = 0.01$

Other type of noise (w/o mathematical reasoning)

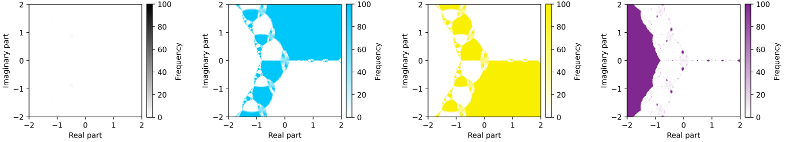


Figure: Real noise where $\lambda_n \sim \mathrm{Unif}([0.975, 1.025])$

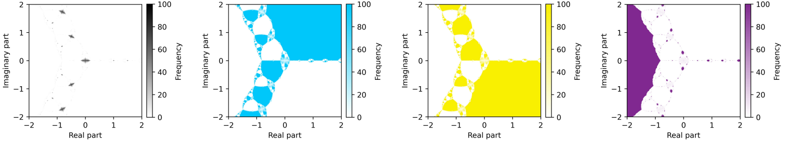


Figure: Real noise where $\lambda_n \sim \mathrm{Unif}([0.975, 1])$

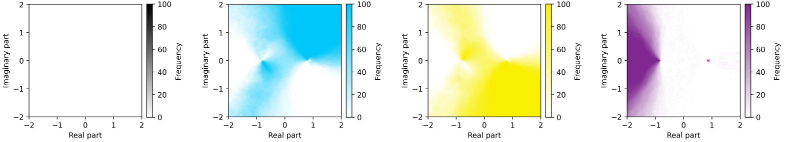


Figure: "Simulated Annealing" where $\lambda_n \sim \mathrm{Unif}(\bar{B}(1, 0.5^n))$

Speed of convergence for other types of noise

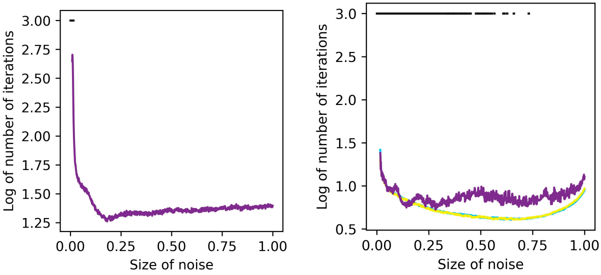


Figure: Left: Uniform distribution on $[1-\rho, 1+\rho]$,
Right: Simulated Annealing where $\lambda_n \sim \mathrm{Unif}(\bar{B}(1, 0.5^n))$.
y-axis represents $\log_{10} \mathbb{E}(T|T<\infty)$ with initial point $z_0 = 0$ and $\epsilon = 0.01$

Remark that there is no mathematical guarantee for global convergence so far.

Another target (1): $Q(z) = z^4 - 6z^2 + \frac{7-8\sqrt{13}}{3}$

The deterministic map $N_{Q,1}$ has two 2-periodic orbits.

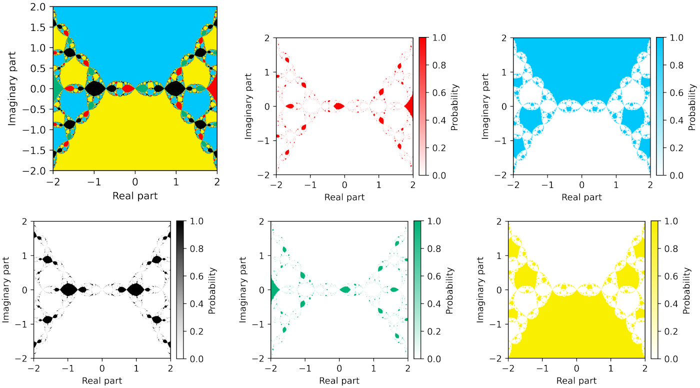


Figure: There is symmetry under complex conjugate.

Another target (2): $Q(z) = z^4 - 6z^2 + \frac{7-8\sqrt{13}}{3}$

I omitted the symmetrical figures.

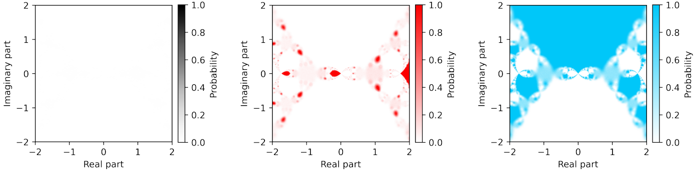


Figure: $\rho = 0.05$ (enough large)

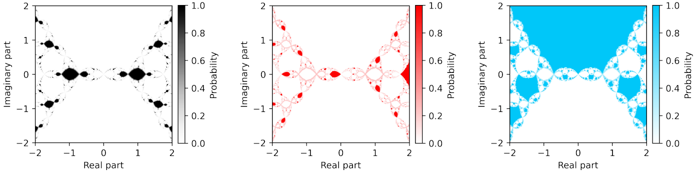
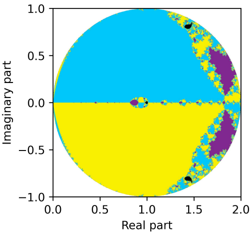


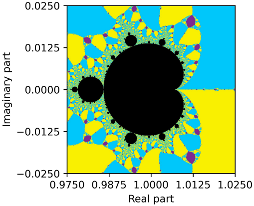
Figure: $\rho = 0.01$ (too small)

A quick summary

1. Deterministic Newton's method can have attractors which do not correspond to the root.
2. Random relaxed Newton's methods succeeds with prob 1 because undesirable attractors are broken when sufficiently large noise is inserted ($\rho > 1/2$ is enough).
3. It may work even with very small noise (e.g. $\rho = 0.01$)

I want to find the universal and smallest constant ρ. The key word might be (deterministic or stochastic) bifurcation.





Numerical observations of Random Relaxed Newton's Methods

Theoretical work on stochastic bifurcation of $\{z \mapsto z^2 + c\}_{c \in \mathbb{C}}$

The family of quadratic polynomials

Newton's maps are rational functions of high degree, but to simplify the discussion we will examine the family of the quadratic polynomials in detail.

$$f_c(z) = z^2 + c \ (c \in \mathbb{C})$$

Why $\{f_c\}_{c \in \mathbb{C}}$?

Theorem 9 (McMullen '00)

"The Mandelbrot set is universal."

Here, the celebrated Mandelbrot set is the bifurcation structure of this $\{f_c\}_{c \in \mathbb{C}}$.

Similarity/difference between $N_{P,\lambda}$ and f_c

The quadratic family $\{f_c\}_c$ is a simplified model of other families. In particular,

	$\{N_{P,\lambda}\}_\lambda$	$\{f_c\}_c$
degree	$\deg P$	2
type of maps	rational	polynomial
common fixed point(s)	roots of P	∞

Let $P(z) = z^3 - 2z + 2$ and consider $c = -1$.

	$N_{P,1}$	f_{-1}
superattracting cycles	$0 \mapsto 1 \mapsto 0$	$0 \mapsto -1 \mapsto 0$

The filled Julia set

Definition 10

For $c \in \mathbb{C}$, define $f_c^{\circ n} = f_c \circ \cdots \circ f_c \circ f_c$ and

$$K_c = \{z_0 \in \mathbb{C} \cup \infty \colon f_c^{\circ n}(z_0) \not\to \infty \ (n \to \infty)\}.$$

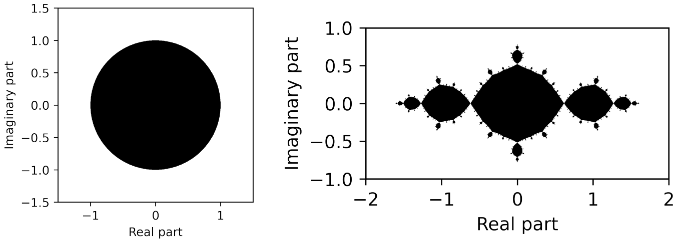


Figure: The black set is K_c $(c = 0, \ -1)$.

We call K_{-1} the basilica.

The Mandelbrot set $\mathcal{M}$

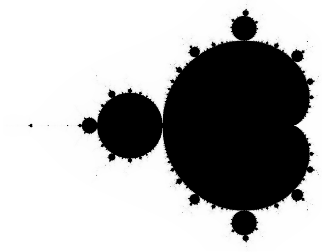
The dynamics is determined by the orbit of $z = 0$.

Theorem 11

The set K_c is connected if and only if the critical orbit is bounded $f_c^{\circ n}(0) \not\to \infty$. Otherwise, K_c is totally disconnected.

Definition 12

$$\mathcal{M} = \{c \in \mathbb{C}\colon K_c \text{ is connected}\} = \{c \in \mathbb{C}\colon f_c^{\circ n}(0) \not\to \infty\}$$



Random Julia sets

Setting 13

Choose iid sequences $c_1, c_2, \ldots$ following the distribution on the disk $\bar{B}(c, \rho) = \{c' \in \mathbb{C} \colon |c' - c| \le \rho\}$.

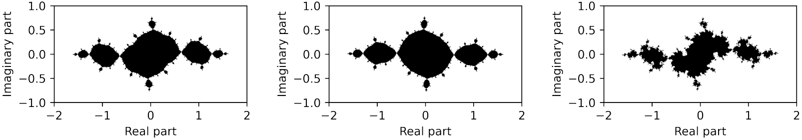
Definition 14

For a sequence $\omega = (c_n)_{n=1}^{\infty} \in \bar{B}(c, \rho)^{\mathbb{N}}$, define

$$f_{\omega}^{(n)} := f_{c_n} \circ \cdots \circ f_{c_2} \circ f_{c_1}.$$

Also, we define its random filled Julia set as follows.

$$K_{\omega} = \{z_0 \in \mathbb{C} \cup \infty \colon f_{\omega}^{(n)}(z_0) \not\to \infty \ (n \to \infty)\}$$



Definition 15 (Fornæss-Sibony '91, Sumi '13)

Suppose that f_c has an attracting cycle in $\mathbb{C}$ (say $c = -1$). Then there exists $r_{\mathrm{bif}}(c) > 0$ such that

- if $0 < \rho < r_{\mathrm{bif}}(c)$, then $\exists T_\infty$ a continuous function s.t. the random orbit $f_{c_n} \circ \cdots \circ f_{c_2} \circ f_{c_1}(z)$

  diverges to ∞ with probability $T_\infty(z)$, and

  converges to $\exists$ a planar attractor with prob $1 - T_\infty(z)$.
- if $\rho > r_{\mathrm{bif}}(c)$, then for every z, with probability 1, $f_{c_n} \circ \cdots \circ f_{c_2} \circ f_{c_1}(z) \to \infty$ $(n \to \infty)$.

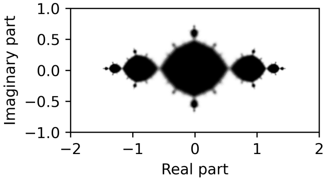


Figure: The function $T_\infty(z)$, the probability of random orbits which converge to ∞ ($\rho < r_{\mathrm{bif}}(c)$ and $\rho > r_{\mathrm{bif}}(c)$)

Main result A: connectedness and bifurcation

Assume a technical condition " $\mathrm{int}\bar{B}(c,\rho)$ contains a superattracting parameter."

Main Result A (W. '22+)

The equivalent statements of the former or the later hold.

1. $\rho \le r_{\mathrm{bif}}(c)$.
2. $\forall \omega \in \bar{B}(c,\rho)^{\mathbb{N}}$, the random Julia set K_ω is connected.
3. $\forall \omega \in \bar{B}(c,\rho)^{\mathbb{N}}$, the critical orbit $f_\omega^{(n)}(0) \not\to \infty$.

$1'$. $\rho > r_{\mathrm{bif}}(c)$.

$2'$. For a.e. $\omega \in \bar{B}(c,\rho)^{\mathbb{N}}$, K_ω is totally disconnected.

$3'$. For a.e. $\omega \in \bar{B}(c,\rho)^{\mathbb{N}}$, the critical orbit $f_\omega^{(n)}(0) \to \infty$.

This gives a natural generalization of deterministic theory.
Recall:
$\mathcal{M} = \{c \in \mathbb{C} \colon K_c \text{ is connected}\} = \{c \in \mathbb{C} \colon f_c^{\circ n}(0) \not\to \infty\}$

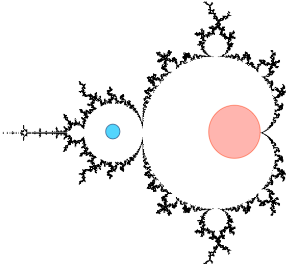
Main result B: quantitative estimates for r_{bif}

Main Result B (W. '22+)

1. For every $c \in \mathbb{C}$ we have $r_{\text{bif}}(c) \leq \text{dist}(c, \partial \mathcal{M})$.
2. If $0 \leq c \leq 1/4$, then $r_{\text{bif}}(c) = 1/4 - c$.
3. If $-1/2 \leq c < 0$, then $r_{\text{bif}}(c) \leq 1/4 - c - c^2$.
4. If $c = -1$, then $0.0386 \cdots \leq r_{\text{bif}}(-1) \leq 0.0399 \cdots$.

2 shows $r_{\text{bif}}(c) = \text{dist}(c, \partial \mathcal{M})$,
3 shows $r_{\text{bif}}(c) < \text{dist}(c, \partial \mathcal{M})$.
In particular,
4 shows $r_{\text{bif}}(c) \ll \text{dist}(c, \partial \mathcal{M})$!

Also, for $c_3 \approx -1.7$
we have $r_{\text{bif}}(c_3) \ll \text{dist}(c_3, \partial \mathcal{M})$.



Conclusions & Discussions

1. The Newton's map $N_{P,1}$ for $P(z) = z^3 - 2z + 2$ has a 2-cycle which does not correspond to any root of P.
2. The randomized root-finding algorithm succeeds if the size of noise ρ satisfies $\rho > 0.01$, and fail if $\rho < 0.005$.
3. This value seems to be related to the "small Mandelbrot set".
4. We can rigorously estimate when the stochastic bifurcation occurs for the quadratic family $f_c(z) = z^2 + c$.
5. Both families (seem to) follow the same mechanism.