Global Dynamics of Mean Curvature Flows

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Mean Curvature Flow

A family of **smooth closed embedded** hypersurfaces $\{M_t\}_{t\in[0,T)}$ in $\mathbb{R}^{n+1}$ is flowing by mean curvature if they satisfy the equation

$$\partial_t x = -H\vec{n}(x).$$

Here H is the mean curvature and $\vec{n}$ is the outer unit normal. The equation is equivalent to

$$\partial_t x = \Delta_{M_t} x.$$

Mean curvature flow (MCF) is a geometric heat flow.

Mean Curvature Flow - examples[1]

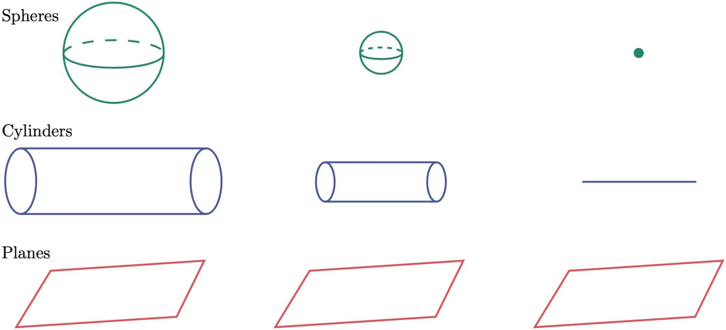


Figure: Sphere, cylinder and plane

[1] Figure from Colding-Minicozzi-Pedersen, BAMS,2015

- There are many similarities between MCF and the Ricci flow.
- After the success of Perelman's solution to the Poincaré conjecture using the Ricci flow, there is an expectation to develop a parallel program for the MCFs.
- Ricci flow concerns the intrinsic geometry, while the MCF concerns the extrinsic geometry.

Curve shortening flow

• A curve shortening flow in $\mathbb{R}^2$ is a flow evolving a simple closed curve γ under the differential equation

$$\frac{d\gamma}{dt} = -k(\gamma)\mathbf{n}(\gamma).$$

This gives the fastest way to decrease the length of the curve.

• It was known classically that any simple closed curve evolved under the CSF will eventually die at a round point. This gives a flow proof of the Jordan curve theorem.

Curve shortening flow[2]

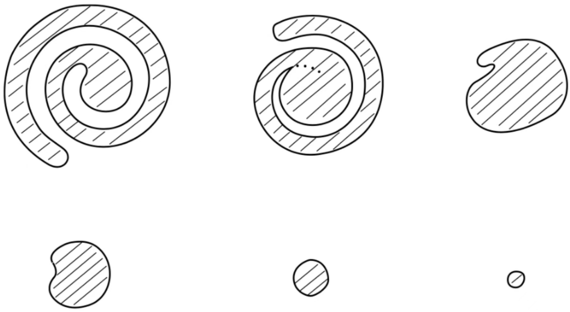


Figure: Curve Shortening Flow

[2]Figure from Colding-Minicozzi-Pedersen, BAMS,2015

Mean Curvature Flow– Dumb bell

However, the MCFs in the higher dimensional case are much more complicated.

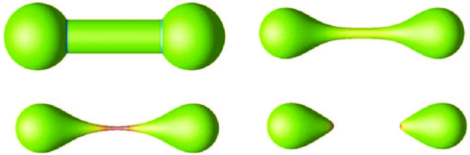


Figure: Dumb bell

Mean Curvature Flow - singularity

A key problem in MCF is to understand the singularity and the singular behaviour. It is standard to blow-up the singularity.

Definition (Huisken 1990)

A **rescaled mean curvature flow (RMCF)** is a family of hypersurfaces satisfying

$$\partial_t x = -\left(H - \frac{\langle x, \vec{\mathbf{n}} \rangle}{2} \right) \vec{\mathbf{n}}.$$

The F-functional

The RMCF is the negative gradient flow of the F-functional

$$F(M) := \int_M e^{-\frac{|x|^2}{4}}\, d\mu.$$

If M_t evolves under RMCF, then we have

$$\frac{d}{dt}F(M_t) = -\int_{M_t}\left(H - \frac{\langle x, \vec{\mathbf{n}}\rangle}{2}\right)^2 e^{-\frac{|x|^2}{4}}\, d\mu.$$

Compare $f:\ \mathbb{R}^2 \to \mathbb{R}$, $\dot{x} = -\nabla f(x)$. Then

$$\frac{d}{dt}f(x(t)) = \nabla f \cdot \dot{x} = -|\nabla f(x)|^2.$$

Definition

A hypersurface is called a **self-shrinker** if it satisfies the equation

$$H - \frac{\langle x, \vec{\mathbf{n}} \rangle}{2} = 0.$$

Self-shrinkers are models of singularities. In fact, rescaled mean curvature flows converge to self-shrinkers (Huisken, Ilmanen, White).

Examples of shrinkers[3]

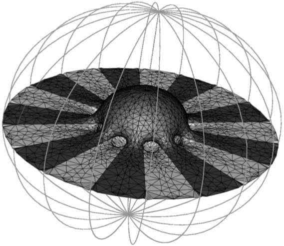
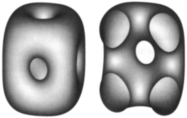
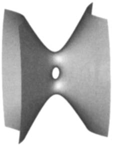


FIGURE 10. A shrinker of the type shown to exist by Kapouleas, Kleene, and Møller in [KKM]; see also Nguyen, [Nu], for a similar shrinker. Its existence had been conjectured by Ilmanen in [I2], where this picture is from. (Used with permission.)



The shrinking cube. Half of the shrinking cube.

A numerical example of Chopp, Exper. Math. 1994.



Chopp, 1994: A self-shrinker asymptotic to a cone.

[3] Figure from Colding-Minicozzi-Pedersen, BAMS,2015

MCF v.s. RMCF

A mean curvature flow $\{\mathcal{M}_\tau\}_{\tau\in[-1,0)}$ corresponds to a rescaled mean curvature flow $\{M_t\}_{t\in[0,\infty)}$ by the relation

$$M_t = e^{t/2}\mathcal{M}_{-e^t}$$

Generic MCFs

If we were able to classify all the self-shrinkers, then we know the singular behaviour very well. However, there are too many self-shrinkers.

It is then reasonable to talk about **generic** mean curvature flows.

This idea was first proposed by Huisken and Angenent-Ilmanen-Chopp.

Why generic dynamics?

Generic: countable intersection of open and sense sets in certain parameter space.

When a system is very complicated and hard to classify, we can in general try to study the generic properties in order to avoid lots of pathological cases and maintain only the **stable** behaviors.

Example: Let A be a $d \times d$ matrix whose leading eigenvalue is positive and simple with eigenvector ϕ_1. Then for generic vector $v (\neq 0) \in \mathbb{R}^d$, the limit

$$\frac{A^n v}{\|A^n v\|} \to \phi_1.$$

Two conjectures on generic MCFs in Ilmanen's list

Huisken conjecture: Generic MCFs in $\mathbb{R}^3$ have only spherical and cylindrical singularities.

Isolatedness conjecture: Generic weak MCFs have only isolated singularities at isolated times.

Dynamical picture

Recall the standard hyperbolic dynamics. Shrinker are fixed points of the RMCF.

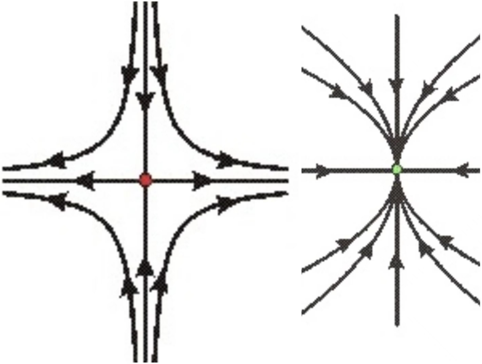
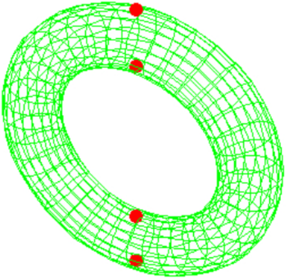


Figure: Unstable and stable fixed points

Huisken conjecture from Morse theoretic viewpoint

- Let $f: \mathbb{T}^2 \to \mathbb{R}$ be a generic Morse function that gives the negative gradient flow $\dot{x} = -\nabla f(x)$.
- Then generic initial condition gives orbit avoiding the sources and hyperbolic fixed points and converging to the stable fixed points.
- Huisken conjecture $\Longrightarrow$ a closed embedded manifold can be decomposed into connected sums of spheres and cylinders.



Dynamical picture

Given a self-shrinker Σ, the **linearized operator** L is defined by

$$Lf = \Delta f - \frac{1}{2}\langle x, \nabla f\rangle + (|A|^2 + 1/2)f.$$

The operator is self-adjoint with respect to the inner product

$$\langle u, v\rangle = \int_\Sigma u(x)v(x)e^{-\frac{|x|^2}{4}}\,d\mu,$$

so all eigenvalues are real, infinitely many negative eigenvalues.

Some known eigenvalues and eigenfunctions:

- Dilation: $LH = H$,
- Translation: $L\langle \mathbf{n}, v\rangle = \frac{1}{2}\langle \mathbf{n}, v\rangle$, for all $v \in \mathbb{R}^{n+1}$.

First eigenfunction and genericity

Observation: the first eigenfunction ϕ_1 of the linearized operator L does not change sign.

Therefore, if the self-shrinker does not have $H > 0$, then H is not the first eigenfunction.

Theorem (Colding-Minicozzi 2012)

The only shrinkers with $H > 0$ are $\mathbb{S}^n(\sqrt{2n})$ and $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$.

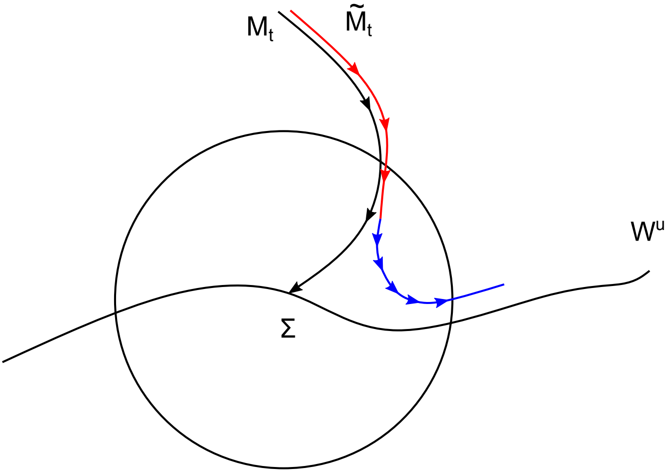
For any shrinker that is not $\mathbb{S}^n$ or $\mathbb{S}^{n-k} \times \mathbb{R}^k$, the leading eigenvalue $\lambda_1 > 1$ and we can perturb towards the direction of ϕ_1 to decrease the value of F-functional, which cannot be remedied by rigid transformations. In this sense, the shrinker is not stable.

Generic dynamics of MCFs

Theorem (Sun-X., Chodosh-Choi-Mantoulidis-Schulze)

If a mean curvature flow $\{\mathcal{M}_t\}$ has a singularity modeled by an unstable self-shrinker Σ that is either compact, or noncompact and asymptotically conical, then we can **generically** *perturb the* **initial** *hypersurface $\mathcal{M}_0$ to avoid this singularity.*

Outline of the proofs



Local dynamics: the blue curve

The flow M_t is very close to the limiting shrinker Σ, so we can write M_t as the graph of a function u over Σ and the MCF equation can be written as

$$\partial_t u = L_\Sigma u + Q(u, \nabla u, D^2 u),$$

where Q is quadratically small in u.
We can approximate the system by the linear system $\partial_t u = L_\Sigma u$.

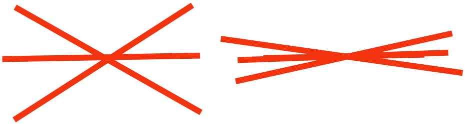


Figure: Action of the cone under the map $\begin{bmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{bmatrix}$

Global dynamics: the red curve

We write the perturbed flow $\tilde{M}_t$ as the graph of a function u_t over the unperturbed flow M_t. The following equation

$$\partial_t u = L_{M_t} u + \mathcal{Q}(t, x, u, \nabla u, \nabla^2 u).$$

measures the difference between the two flows.

The linear part

$$\partial_t u = L_{M_t} u$$

is called the **variational equation**, similar to the Jacobi field equation or tidal force in physics.

We use the variational equation to control the red segment.

Main difficulty

The main difficulty: how to send an initial perturbation into a cone around ϕ_1 using the variational equation?

The first eigenfunction $\phi_1 > 0$. Thus u has a definite L^2-projection to ϕ_1 direction, i.e.

$$\langle u, \phi_1 \rangle > C\|u\|_{L^2}.$$

This shows that u enters a fixed invariant cone.

The compact shrinker case: Li-Yau estimate

Theorem (Li-Yau's Harnack inequality)

Suppose $v > 0$ *satisfies* $\partial_t v = L_{M_t} v$ *on* $\{M_t\}$*. There exist* T_0, τ, C $t_2 > t_1 > T$*, and* $t_2 - t_1 > \tau$*, we have for all* $x \in M_{t_1}, y \in M_{t_2}$

$$v(x, t_1) \leq Cv(y, t_2).$$

The compact shrinker case: Li-Yau estimate

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$$v(x, t_1) \leq Cv(y, t_2).$$

Theorem (Harnack estimate on the same time slice)

Suppose $v > 0$ satisfies $\partial_t v = L_{M_t} v$ on $\{M_t\}$. There exist T, C such that for all $t > T$ we have

$$\max v(\cdot, t) \leq C \min v(\cdot, t).$$

The approach to the isolatedness conjecture

- Huisken conjecture $\Longrightarrow$ generic MCF has only spherical and cylindrical singularities.
- Assume Huisken conjecture, then generic isolatedness conjecture is reduced to spherical and cylindrical cases.

Cylindrical singularities

There can be a continuum of cylindrical singularities. One example is the marriage ring.

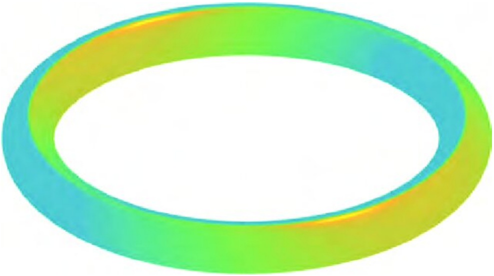

Figure: The marriage ring

Eigenvalues and eigenfunctions of L_{Σ^k}, $\Sigma^k = \mathbb{S}^{n-k} \times \mathbb{R}^k$

eigenvalues of $L_{\mathcal{C}_k}$	corresponding eigenfunctions
1	1
$1/2$	θ_i, y_j
0	$\theta_i y_j, y_i^2 - 2$

Remark

- *The eigenfunction 1 represents infinitesimal dilations.*
- *The eigenfunction θ_i (respectively y_j) represents infinitesimal translations in the $\mathbb{S}^{n-k}$ (resplectively $\mathbb{R}^k$) direction.*
- *The eigenfunction $\theta_i y_j$ represents infinitesimal rotations.*

The mechanism for isolatedness: the normal form

Theorem (the normal form)

Assume that the MCF has a cylindrical singularity modeled on $\Sigma^k = \mathbb{S}^{n-k} \times \mathbb{R}^k$. Then there exist $K > 0$ and $T > 0$ such that for $t > T$, the RMCF is a graph over $\Sigma^k \cap B_{K\sqrt{t}}$, and the graphical function $u(\cdot, t) : \Sigma^k \cap B_{K\sqrt{t}} \to \mathbb{R}$ of $M_t \cap B_{K\sqrt{t}}$ satisfies the following in the (weighted) H^1 normal form up to a rotation in $\mathbb{R}^k$

$$u(z,t) = \sum_{i \in \mathcal{I}} \frac{\sqrt{2(n-k)}}{4t}(y_i^2 - 2) + O(1/t^2),$$

as $t \to \infty$, where $\mathcal{I} \subset \{1, 2, \ldots, k\}$.

Definition

We say the singularity is **nondegenerate**, If the set of indices $\mathcal{I}$ equals $\{1, 2, \ldots, k\}$ and otherwise **degenerate.**

The local theorems

Theorem (Sun-X., the stability theorem)

A nondegenerate cylindrical singularity is stable. In other words, if an MCF has a nondegenerate cylindrical singularity at the spacetime $(0, 0)$*, then any small initial perturbation of it has also a nondegenerate cylindrical singularity close to* $(0, 0)$*.*

Theorem (Sun-X., the denseness theorem)

Nondegenerate cylindrical singularities are dense. In other words, if an MCF has a degenerate cylindrical singularity at the spacetime $(0, 0)$*, then there exists arbitrarily small initial perturbation that creates a nondegenerate cylindrical singularity close to* $(0, 0)$*.*

The global theorem

Theorem (Sun-X.)

Suppose $\{\mathcal{M}_\tau\}_{\tau\in[-1,0)}$ *be an MCF in* $\mathbb{R}^{n+1}$ *and the singular set at time* 0 *is a smooth closed curve* γ *modeled on cylinder* $\mathbb{S}^{n-1}\times\mathbb{R}$*. Then for any* $p\in\gamma$ *and any* $\epsilon_0>0$ *there exists a function* $u_0\in C^{2,\alpha}(\mathcal{M}_{-1})$ *with* $\|u_0\|_{C^{2,\alpha}}<\epsilon_0$ *such that the perturbed MCF* $\{\widetilde{\mathcal{M}}_\tau\}$ *starting from* $\widetilde{\mathcal{M}}_{-1}:=\{x+u_0(x)\mathbf{n}(x):x\in\mathcal{M}_{-1}\}$ *has a single first-time singularity in an* ε_0*-neighborhood of* p*.*

This applies in particular to the example of marriage ring.

The main difficulty: eliminating the y_i mode

- We wish to have a nondegenerate normal form. However, since h_2 is not the leading eigenfunction, it is most likely that the perturbed flow has an exponentially growing mode such as $1, \theta_i, y_j$.
- We can use an Euclidean rigid transformation to eliminate the Fourier modes $1, \theta_i, \theta_i y_j$, but not y_i in general.
- The key observation is that with a $h_2(y_i) = (y_i^2 - 2)$-mode, we can use a translation to eliminate a small y_i mode.

$$(y_i - a)^2 - 2 = (y_i^2 - 2) - 2ay_i + a^2.$$

- This means if there is a y_i component and a nontrivial $h_2(y_i)$ mode, we can make a translation in the y_i direction to eliminate y_i paying the price of introducing a dilation a^2.

The End

Thank you for your attention!