# Some Recent Results on Nonintegrability of Dynamical Systems


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## References


[1] Y, Nonintegrability of nearly integrable dynamical systems near resonant periodic orbits, *J. Nonlinear Sci.*, **32** (2022), 4.

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[2] Y, Non-integrability of the restricted three-body problem, *Ergodic Theory Dynam. Systems*, in press.

Other applications:

[3] Shoya Motonaga and Y, Nonintegrability of forced nonlinear oscillators, *Jpn. J. Ind. Appl. Math.*, **41** (2024), 151–164.

[4] Y, Nonintegrability of dissipative planar systems, *Phys. D*, **461** (2024), 134106.

[5] Y, A new proof of Poincaré's result on the restricted three-body problem, submitted for publication, arXiv:2111.11031.

# Plan of the talk

- Definition of integrability for general dynamical systems (including non-Hamiltonian systems)
- Classical results of Poincaré and Kozlov
- Modern theory for determining the nonintegrability: **Morales-Ramis theory** based on **differential Galois theory**

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- Restricted three-body problem
- Time-periodic perturbations of single-degree-of-freedom Hamiltonian systems
- Related results

## Definition of Integrability for General Dynamical Systems

# Liouville integrability

Consider $n$-degree-of-freedom Hamiltonian system

$$\dot{x} = J_n \mathrm{D}H(x), \quad J_n = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}, \quad x \in \mathbb{R}^{2n}, \qquad \text{(HS)}$$

where $H : \mathbb{R}^{2n} \to \mathbb{R}$ and $\mathrm{id}_n$ is the $n \times n$ identity matrix.

**Definition 1 (Liouville)**
(HS) is called **integrable** if there exist $n$ scalar-valued functions $F_1(x)(:= H(x)), F_2(x), \ldots, F_n(x)$ s.t.

**(i)** $\mathrm{D}F_1(x), \ldots, \mathrm{D}F_n(x)$ are linearly independent almost everywhere (a.e) and the Poisson brackets are zero:

$$\{F_j, F_k\}(x) := \mathrm{D}F_j(x) \cdot J_n \mathrm{D}F_k(x) \equiv 0, \quad j, k = 1, \ldots, n.$$

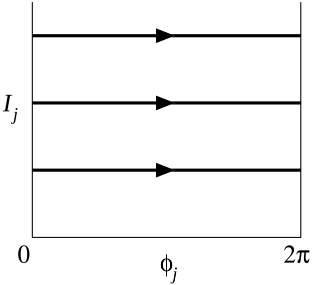
The functions $F_1(x), F_2(x), \ldots, F_n(x)$ are called **first integrals**.

## Remark 1

**(i)** When $n = 1$, (HS) is always integrable.

**(ii)** If the level set $\{x \in \mathbb{R}^{2n} | F_j(x) = c_j, j = 1, \ldots, n\}$, where $c_j$, $j = 1, \ldots, n$, are const., is compact in $\mathbb{R}^{2n}$, then there exist **action-angle coordinates** $(I_j, \phi_j) \in \mathbb{R} \times \mathbb{S}^1$, $j = 1, \ldots, n$, s.t. $H = H(I_1, \ldots, I_n)$:

$$\dot{I}_j = 0, \quad \dot{\phi}_j = \frac{\partial H}{\partial I_j}(I_1, \ldots, I_n), \quad j = 1, \ldots, n.$$



# Bogoyavlenskij Integarbility

Consider a general $n$-dimensional system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \tag{GS}$$

**Definition 2 (Bogoyavlenskij '98)**

(GS) is called **$(q, n-q)$-integrable** if there exist $q$ vector fields $f_1(x)$ $(:= f(x)), f_2(x), \dots, f_q(x)$ and $n-q$ scalar-valued functions $F_1(x)$, $\dots, F_{n-q}(x)$ s.t.

**(i)** $f_1(x), \dots, f_q(x)$ are linearly independent a.e. and commutative, i.e.,

$$[f_j, f_k](x) := \mathrm{D}f_k(x)f_j(x) - \mathrm{D}f_j(x)f_k(x) \equiv 0;$$

**(ii)** $\mathrm{D}F_1(x), \dots, \mathrm{D}F_{n-q}(x)$ are linearly independent a.e. and $F_1(x)$, $\dots, F_{n-q}(x)$ are first integrals of $f_1, \dots, f_q$, i.e.,

$$\mathrm{D}F_k(x) \cdot f_j(x) \equiv 0.$$

## Remark 2

**(i)** If (HS) is Liouville-integrable, then it is $(n, n)$-integrable in the sense of Bogoyavlenskij. Actually, $J_n \mathrm{D}F_1(x), \dots, J_n \mathrm{D}F_n(x)$ are commutative vector fields. Thus, Bogoyavlenskij-integrability is a generalization of Liouville-integrability.

**(ii)** If (GS) is integrable and the level set

$$\{x \in \mathbb{R}^{2n} \mid F_j(x) = c_j, j = 1, \dots, n-q\},$$

where $c_j$, $j = 1, \dots, n-q$, are constants, is compact in $\mathbb{R}^{2n}$, then there exist **action-angle coordinates** $(I_1, \dots, I_{n-q}, \phi_1, \dots, \phi_q) \in \mathbb{R}^{n-q} \times \mathbb{T}^q$ with $\mathbb{T}^q = \prod_{j=1}^q \mathbb{S}^1$ s.t.

$$\dot{I}_j = 0, \quad \dot{\phi}_k = \Omega_k(I_1, \dots, I_{n-q}),$$
$$j = 1, \dots, n-q. \quad k = 1, \dots, q.$$

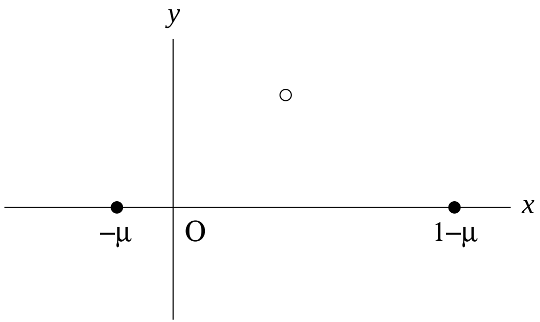
**(iii)** An $n$-dimensional linear system with $f(x) = Ax$, $A \in \mathbb{R}^{n \times n}$, is $(n, 0)$-integrable. For instance, if $A$ is diagonal, then $f_1(x) = Ax$ and $f_j(x) = x_j e_j$, $j = 2, \dots, n$, where $e_1, \dots, e_n \in \mathbb{R}^n$ are the standard basis, are commutative vector fields.

# Classical Results of Poincaré and Kozlov

# Restricted three-body problem (planar case)

$$\dot{x} = p_x + y, \quad \dot{p}_x = p_y + \frac{\partial U_2}{\partial x}(x, y),$$
$$\dot{y} = p_y - x, \quad \dot{p}_y = -p_x + \frac{\partial U_2}{\partial y}(x, y), \qquad \text{(R3BP)}$$

where $U_2(x, y) = \dfrac{\mu}{\sqrt{(x - 1 + \mu)^2 + y^2}} + \dfrac{1 - \mu}{\sqrt{(x + \mu)^2 + y^2}}.$



- Poincaré (1890,1892) showed the nonexistence of a first integral which is analytic in $\mu$ as well as the state variables $x, y, p_x, p_y$ and functionally independent of the Hamiltonian.

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- He considered Hamiltonian systems of the form

$$\dot{I} = -\varepsilon \mathrm{D}_\theta H_1(I, \theta; \varepsilon),\ \dot{\theta} = \mathrm{D}H_0(I) + \varepsilon \mathrm{D}_I H_1(I, \theta; \varepsilon), \quad \text{(AAH)}$$

where $(I, \theta) \in \mathbb{R}^n \times \mathbb{T}^n$, $0 < |\varepsilon| \ll 1$ and $H_0, H_1$ are analytic in all the arguments.

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  where $(I,\theta) \in \mathbb{R}^n \times \mathbb{T}^n$, $0 < |\varepsilon| \ll 1$ and $H_0, H_1$ are analytic in all the arguments.
- Recall that
  - (AA) is integrable when $\varepsilon = 0$;
  - Integrable Hamiltonian systems can generally be transformed into (AA) with $\varepsilon = 0$ if the level set of first integrals is compact.

- **Poincaré set** $\mathscr{P} = \{I \in \mathbb{R}^n \mid r \cdot \mathrm{D}_I H_0(I) = 0, \hat{h}_r(I) \neq 0\}$, where

$$\mathrm{D}_I H_1(I, \theta; 0) = \sum_{r \in \mathbb{Z}^m} \hat{h}_r(I) \exp(ir \cdot \theta).$$

**Theorem 1 (Poincaré 1892, Kozlov '83)**

Suppose that $r \cdot \mathrm{D}_I H_0(I) \equiv 0$ only for $r = 0$, $\mathrm{D}_I H_0(I_0) \neq 0$ for some $I_0 \in \mathbb{R}^n$ and $\mathscr{P}$ is dense in a nbhd of $I_0$. Then (AA) has no first integral which is real-analytic in $(I, \theta, \varepsilon)$ and functionally independent of the Hamiltonian $H_0(I) + \varepsilon H_1(I.\theta; \varepsilon)$.

**Remark 3**

(i) Poincaré applied his original version of Theorem 1 to (R3BP) with $\varepsilon = \mu$. In particular, it was very hard to check that $H_1(I, \theta; \varepsilon)$ satisfies its hypothesis: He used 66 pages for it in the famous monograph published in 1892!

(ii) Theorem 1 does not say about the possibility that (AA) is integrable for a specific value of $\varepsilon$.

(iii) Kozlov's result contains more and was extended to a non-Hamiltonian case in his book published in '96.

## Modern Theory for Determining the Nonintegrability:

## Morales-Ramis Theory Based on the Differential Galois Theory

## Quick review of differential Galois theory

Consider a linear system on a (Riemann) surface $\mathbf{\Gamma}$ (e.g.,a region in $\mathbb{C}$):

$$\dot{x} = A(t)x, \quad x \in \mathbb{C}^n, \quad t \in \mathbf{\Gamma}, \quad A_{ij}(t) \in \mathbb{K}, \tag{LS}$$

where $\mathbb{K}$ is a differential field (i.e., a field endowed with differentiation).

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- $\Phi(t)$: Fundamental matrix (of solutions) to (LS).
- $\mathbb{L} \supset \mathbb{K}$: Differential field extension s.t. $\Phi_{ij}(t) \in \mathbb{L}$, which is called the **Picard-Vessiot extension** of (LS).

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  $\overset{\text{def}}{\Leftrightarrow}$ field automorphism of $\mathbb{L}$ s.t. $\sigma|_{\mathbb{K}} = \mathrm{id}$ and $\sigma \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sigma$.

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- $\mathrm{Gal}(\mathbb{L}/\mathbb{K}) = \{M_\sigma \mid \sigma \text{ is a } \mathbb{K}\text{-automorphism of } \mathbb{L}\}$:
  **differential Galois group** of (LS)
- An algebraic group such as $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ has a connected component containing the identity, which is called the **identity component**.

- If the identity component of $\mathbf{Gal}(\mathbb{L}/\mathbb{K})$ is conjugate to a triangular group, then (LS) is solved by quadrature.

## Simple example

$$\dot{x} = \begin{pmatrix} 2t & 0 \\ 2t & 2t \end{pmatrix} x, \quad x \in \mathbb{C}^2, \quad t \in \Gamma = \mathbb{C}.$$

- $\mathbb{K} = \mathbb{C}(t)$, which consists of all rational functions of $t$.
- $x = \begin{pmatrix} e^{t^2} \\ t^2 e^{t^2} \end{pmatrix}, \begin{pmatrix} 0 \\ e^{t^2} \end{pmatrix}$ are linearly independent solutions.
- $\Phi(t) = \begin{pmatrix} e^{t^2} & 0 \\ t^2 e^{t^2} & e^{t^2} \end{pmatrix}$ is a fundamental matrix.
- $\mathbb{L} = \mathbb{C}(t, e^{t^2})$, which consists of all rational functions $t$ and $e^{t^2}$.
- For $\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})$,

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}\sigma(e^{t^2})}{\sigma(e^{t^2})} = \sigma\left(\frac{\frac{\mathrm{d}}{\mathrm{d}t}e^{t^2}}{e^{t^2}}\right) = \sigma(2t) = 2t.$$

$\therefore \log|\sigma(e^{t^2})| = \log|e^{t^2}| + C_0$ for some $C_0 \in \mathbb{C}$.

$\therefore \sigma(e^{t^2}) = Ce^{t^2}$ with $C = e^{\pm C_0} \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

- $\sigma(\Phi(t)) = \begin{pmatrix} Ce^{t^2} & 0 \\ Ct^2e^{t^2} & Ce^{t^2} \end{pmatrix} = \Phi(t) \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.
- $\mathrm{Gal}(\mathbb{L}/\mathbb{K}) = \left\{ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \middle| C \in \mathbb{C}^* \right\}$.

# Morales-Ramis theory (extension by Ayoul & Zung)

General system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n. \tag{GS}$$

- $x = \phi(t)$: nonconstant solution
- **Variational equation (VE)** along $x = \phi(t)$:

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**Theorem 2 (Morales-Ruiz & Ramis 2001, Ayoul & Zung, 2010)**
If (GS) is meromorphically integrable near $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{t})$, then the identity component of the **differential Galois group** is commutative.

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If (GS) is meromorphically integrable near $x = \phi(t)$, then the identity component of the **differential Galois group** is commutative.

Hence, the identity component of the differential Galois group is not commutative, then (GS) is not meromorphically integrable near $x = \phi(t)$,

**Remark 4.**

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**(iv)** It is an important fact in application of the Morales-Ramis theory that the identity component of the differential Galois group may be triangularizable, the corresponding linear system is solved by quadrature, even it is not commutative.

# 3. Nonintegrability of Nearly Integrable Systems near Periodic Orbits

## Nearly integrable systems

Action-angle coordinates

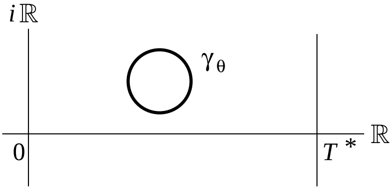
$$\dot{I} = \varepsilon h(I,\theta;\varepsilon),\ \dot{\theta} = \omega(I) + \varepsilon g(I,\theta;\varepsilon),\ (I,\theta) \in \mathbb{R}^\ell \times \mathbb{T}^m \quad \text{(AA)}$$

**(A1)** For some $I^* \in \mathbb{R}^\ell$, $\dim_{\mathbb{Q}}\langle \omega_1(I^*), \ldots, \omega_m(I^*)\rangle = 1$, i.e.,

$$^\exists\omega^* > 0 \quad \text{s.t.} \quad \omega(I^*)/\omega^* \in \mathbb{Z}^m \setminus \{0\}.$$

**(A2)** For some $k \geq 0$, there exists a closed loop $\gamma_\theta$ for some $\theta \in \mathbb{T}^m$ s.t.

$$\mathscr{I}^k(\theta) := \mathrm{D}\omega(I^*) \int_{\gamma_\theta} \mathrm{D}_\varepsilon^k h(I^*, \omega(I^*)\tau + \theta; 0)\mathrm{d}\tau \neq 0.$$




$$T^* = 2\pi/\omega^*$$

- $\mathscr{T}^* = \{(\boldsymbol{I}^*, \boldsymbol{\theta}) \mid \boldsymbol{\theta} \in \mathbb{T}^m\}$: **resonant torus**.
- $(\boldsymbol{I}, \boldsymbol{\theta}) = (\boldsymbol{I}^*, \boldsymbol{\omega}(\boldsymbol{I}^*)t + \boldsymbol{\theta_0})$ on $\mathscr{T}^*$ for each $\boldsymbol{\theta_0} \in \mathbb{T}^m$: **resonant periodic orbit**.

**Theorem 3 (Y).** Let $\boldsymbol{D}$ be any domain in $\mathbb{C}/\boldsymbol{T}^*\mathbb{Z}$ containing $\mathbb{R}/\boldsymbol{T}^*\mathbb{Z}$ and $\boldsymbol{\gamma_\theta}$. Under (A1) and (A2), (AA) is not meromorphically B-integrable near $(\boldsymbol{I}^*, \boldsymbol{\omega}(\boldsymbol{I}^*)\boldsymbol{\tau} + \boldsymbol{\theta})$ with $\boldsymbol{\tau} \in \boldsymbol{D}$ s.t. the first integrals and commutative vector fields also depend meromorphically on $\varepsilon$ near $\varepsilon = \mathbf{0}$. Moreover, if (A2) holds for $\boldsymbol{\theta} \in \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ is a dense set of $\mathbb{T}^m$, then the conclusion holds for any resonant periodic orbit on $\mathscr{T}^*$.

**Remark 5.** When (AA) is Hamiltonian, it is not meromorphically L-integrable if the hypotheses of Theorem 3 hold.

# Sketch of the proof (for $k = 0$)

Extend (AA) as

$$\dot{I} = \varepsilon h(I, \theta; \varepsilon), \quad \dot{\theta} = \omega(I) + \varepsilon g(I, \theta; \varepsilon), \quad \dot{\varepsilon} = 0 \qquad \text{(AAE)}$$

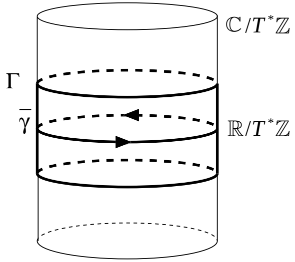
Variational equation (VE) along $(I, \theta, \varepsilon) = (I^*, \omega(I^*)t + \theta_0, 0)$:

$$\begin{aligned} &\dot{\xi} = h(I^*, \omega^* t + \theta_0; 0)\chi, \\ &\dot{\eta} = \mathrm{D}\omega(I^*)\xi + g(I^*, \omega^* t + \theta_0; 0)\chi, \quad (\xi, \eta, \varepsilon) \in \mathbb{C}^\ell \times \mathbb{C}^m \times \mathbb{C}, \\ &\dot{\chi} = 0, \end{aligned}$$

which is regarded as a linear system over $\mathbb{K}_\theta$ on the Riemann surface $\Gamma$, where $\mathbb{K}_\theta \neq \mathbb{C}$ is a differential field that contains the elements of

$$h(I^*, \omega(I^*)t + \theta; \varepsilon),$$
$$g(I^*, \omega(I^*)t + \theta; \varepsilon),$$

where $T^* = 2\pi/\omega^*$.



Fundamental matrix of VE:

$$\Phi(t;\theta_0)=\begin{pmatrix}\mathrm{id}_\ell & 0 & \Xi(t;\theta_0)\\ \mathrm{D}\omega(I^*)t & \mathrm{id}_m & \Psi(t;\theta_0)\\ 0 & 0 & 1\end{pmatrix},$$

$$\Xi(t;\theta)=\int_0^t h(I^*,\omega(I^*)\tau+\theta)\mathrm{d}\tau,$$

$$\Psi(t;\theta)=\int_0^t \big(\mathrm{D}\omega(I^*)\Xi(\tau;\theta)+g(I^*,\omega(I^*)\tau+\theta;0)\big)\mathrm{d}\tau.$$

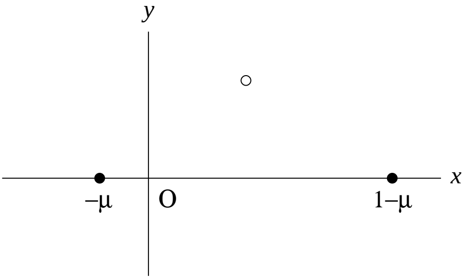
Applying Theorem 2, we obtain the desired result.
For $k\geq 1$, we use the Morales-Ramis-Simó theory. □

# Restricted Three-Body Problem

# Restricted three-body problem (for the planar case)

$$\dot{x} = p_x + y, \quad \dot{p}_x = p_y + \frac{\partial U_2}{\partial x}(x, y),$$
$$\dot{y} = p_y - x, \quad \dot{p}_y = -p_x + \frac{\partial U_2}{\partial y}(x, y), \qquad \text{(R3BP)}$$

where $U_2(x, y) = \dfrac{\mu}{\sqrt{(x - 1 + \mu)^2 + y^2}} + \dfrac{1 - \mu}{\sqrt{(x + \mu)^2 + y^2}}$.




**Theorem 4 (Poincaré).** (R3BP) is not analytically integrable s.t. the first integrals depend analytically on $\mu$ near $\mu = 0$.

Regard (R3BP) as a Hamiltonian system on

$$\mathscr{S}_2 =\{(x, y, p_x, p_y, u_1, u_2) \in \mathbb{C}^6 \\ \mid u_1^2 - (x - 1 + \mu)^2 - y^2 = u_2 - (x + \mu)^2 - y^2 = 0\}.$$

It is written as a meromorphic (rational) system

$$\begin{aligned} &\dot{x} = p_x + y, \quad \dot{y} = p_y - x, \\ &\dot{p}_x = p_y - \mu(x - 1 + \mu)/u_1^3 - (1 - \mu)(x + \mu)/u_2^2, \\ &\dot{p}_y = -p_x - \mu y/u_1^3 - (1 - \mu)y/u_2^2, \\ &\dot{u}_1 = ((x - 1 + \mu)(p_x + y) + y(p_y - x))/u_1, \\ &\dot{u}_2 = ((x + \mu)(p_x + y) + y(p_y - x))/u_2. \end{aligned}$$

**Theorem 5 (Y).** The problem (R3BP) is meromorphically nonintegrable in punctured neighborhoods of $(x, y) = (-\mu, 0)$ and $(1 - \mu, 0)$ for any $\mu \in (0, 1)$.

## Sketch of the proof

- Consider a nbhd of $(x, y) = (-\mu, 0)$.
- Let $\varepsilon^2\xi = x + \mu$, $\varepsilon^2\eta = y$, $\varepsilon^{-1}p_\xi = p_x$, $\varepsilon^{-1}p_\eta = p_y + \mu$.
- After scaling $t \to t/\varepsilon^3$, up to the order of $\varepsilon^6$,

$$\dot{\xi} = p_\xi + \varepsilon^3\eta, \quad \dot{p}_\xi = -\frac{(1-\mu)\xi}{(\xi^2+\eta^2)^{3/2}} + \varepsilon^3 p_\eta + 2\varepsilon^6\mu\xi,$$

$$\dot{\eta} = p_\eta - \varepsilon^3\xi, \quad \dot{p}_\eta = -\frac{(1-\mu)\eta}{(\xi^2+\eta^2)^{3/2}} - \varepsilon^3 p_\xi - \varepsilon^6\mu\eta.$$

- Hamiltonian

$$H = \tfrac{1}{2}(p_\xi^2 + p_\eta^2) - \frac{1-\mu}{\sqrt{\xi^2+\eta^2}} + \varepsilon^3(\eta p_\xi - \xi p_\eta) - \tfrac{1}{2}\varepsilon^6\mu(2\xi^2 - \eta^2).$$

- Using Delaunay elements, we rewrite the above system in action-angle coordinates.
- Application of Theorem 3.1 yields the desired result. □

**Remark 6.** Similarly, we can prove Poincaré's result (Theorem 4).

# Time-Periodic Perturbations of Single-Degree-of-Freedom Hamiltonian Systems

## Time-periodic perturbations of s.d.o.f. Hamiltonian system

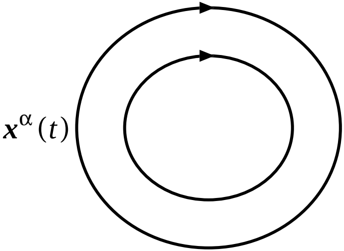
$$\dot{x} = J\mathrm{D}H(x) + \varepsilon u(x, \nu t), \quad x \in \mathbb{R}^2, \tag{TPP}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

**(M1)** When $\varepsilon = 0$, there exists a one-parameter family of periodic orbits $x^\alpha(t)$, $\alpha \in (\alpha_1, \alpha_2)$, with period $T^\alpha > 0$ for some $\alpha_1 < \alpha_2$;

**(M2)** $x^\alpha(t)$ is analytic with respect to $\alpha \in (\alpha_1, \alpha_2)$.



Action variable:

$$I^\alpha = \frac{1}{2\pi}\int_{x^\alpha} x_2 \mathrm{d}x_1 = \frac{1}{2\pi}\int_0^{T^\alpha} x_2^\alpha(t)\dot{x}_1^\alpha(t)\mathrm{d}t$$

Symplectic transformation:

$$x = x^{\alpha(I)}\left(\frac{\theta_1}{\Omega(I)}\right), \quad \Omega(I) = \frac{2\pi}{T^{\alpha(I)}}.$$

Action-angle coordinates:

$$\dot{I} = \varepsilon h(I,\theta_1,\theta_2), \quad \dot{\theta}_1 = \Omega(I) + \varepsilon g_1(I,\theta_1,\theta_2), \quad \dot{\theta}_2 = \nu,$$

where

$$h(I,\theta_1,\theta_2) = \frac{1}{\Omega(I)}\mathrm{D}H\left(x^{\alpha(I)}\left(\frac{\theta_1}{\Omega(I)}\right)\right)\cdot u\left(x^{\alpha(I)}\left(\frac{\theta_1}{\Omega(I)}\right),\theta_2\right),$$

$$g_1(I,\theta_1,\theta_2) = J\frac{\partial}{\partial I}x^{\alpha(I)}\left(\frac{\theta_1}{\Omega(I)}\right)\cdot u\left(x^{\alpha(I)}\left(\frac{\theta_1}{\Omega(I)}\right),\theta_2\right).$$

- At $\alpha = \alpha^{l/n}$, $2\pi/T^\alpha = n\nu/l$ for $l, n > 0$ relatively prime integers.
- (A1) holds with $\omega^* = 2\pi/nT^\alpha = \nu/l$.
- **Subharmonic Melnikov function**:

$$M^{l/n}(\phi) = \int_0^{2\pi l/\nu} \mathrm{D}H(x^\alpha(t)) \cdot u(x^\alpha(t), \nu t + \phi)\mathrm{d}t.$$

- $M^{l/n}(\phi)$ has a simple zero at $\phi = \phi_0$ and $\mathrm{d}T^\alpha/\mathrm{d}\alpha \neq 0$
$\Rightarrow$ $^\exists$periodic orbit near $(x, \phi) = (x^\alpha(t), \nu t + \phi_0)$ in (TPP).

**Theorem 6 (Y).** At $\alpha = \alpha^{l/n}$, $\mathrm{d}T^\alpha/\mathrm{d}\alpha \neq 0$ and $^\exists\gamma_\phi$ for some $\phi \in \mathbb{S}^1$ s.t.

$$\hat{\mathscr{I}}(\phi) := \int_{\gamma_\phi} \mathrm{D}H(x^\alpha(t)) \cdot u\left(x^\alpha(\tau), \nu\tau + \phi\right) \mathrm{d}\tau \neq 0.$$

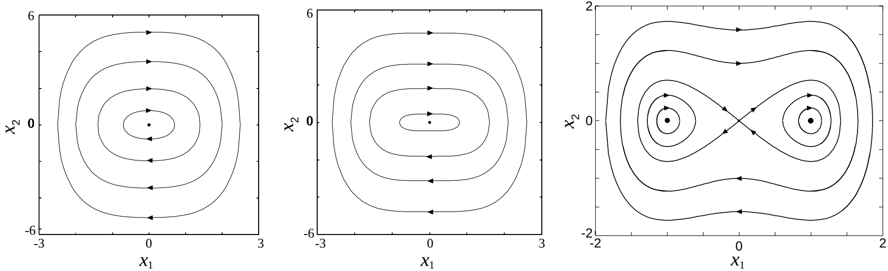
$\Rightarrow$ (TPP) is not meromorphically integrable in the meaning of Theorem 3 near the resonant periodic orbit $(x^\alpha(\tau), \nu\tau + \phi)$ with $\alpha = \alpha^{l/n}$ on any domain $\hat{\Gamma}$ in $\mathbb{C}/(2\pi l/\nu)\mathbb{Z}$ containing $\mathbb{R}/(2\pi l/\nu)\mathbb{Z}$ and $\hat{\gamma}_\phi$.

## Other applications

### Duffing Oscillators:

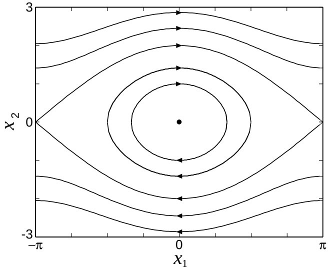
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_1 - x_1^3 + \varepsilon(\beta \cos \nu t - \delta x_2), \qquad \text{(DO)}$$

where $a = 1, 0, -1$.



**Forced pendulum**:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 + \varepsilon(\beta \cos \nu t - \delta x_2). \qquad \text{(FP)}$$




**Remark 7.** Motonaga and Y (2023,2024) showed that (DO) with $a = -1$ and (FP) are not real-analytically integrable in the meaning of Theorem 3 near homoclinic orbits. Moreover, Motonaga (2024) also obtained similar results for resonant periodic orbits.

# Related Results

- Real-analytic nonintegrability: More direct extension of Poincaré and Kozlov to general systems:
  Motonaga and Y, Obstructions to integrability of nearly integrable dynamical systems near regular level sets, *Arch. Ration. Mech. Anal.*, **247** (2023), 44.
  [**3**] Motonaga and Y, Nonintegrability of forced nonlinear oscillators, *Jpn. J. Ind. Appl. Math.*, **41** (2024), 151–164.

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- Real-meromorphic nonintegrability of periodic perturbations of s.d.o.f Hamiltonian systems near homo- and heteroclinic orbits: Y, Nonintegrability of time-periodic perturbations of single-degree-of-freedom Hamiltonian systems near homo- and heteroclinic orbits, *Physi. D* (2024), **464**, 134189.

- Real-analytic nonintegrability: More direct extension of Poincaré and Kozlov to general systems:
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- Real-meromorphic nonintegrability of periodic perturbations of s.d.o.f Hamiltonian systems near homo- and heteroclinic orbits: Y, Nonintegrability of time-periodic perturbations of single-degree-of-freedom Hamiltonian systems near homo- and heteroclinic orbits, *Physi. D* (2024), **464**, 134189.
- Normal forms of codimension-two (fold-Hopf & double Hopf) bifurcations:
  Y, Nonintegrability of the unfolding of the fold-Hopf bifurcation, *Nonlinearity*, **31** (2018), 341-350.
  Acosta-Humánez & Y, Nonintegrability of the unfoldings of codimension-two bifurcations, *Nonlinearity*, **31** (2020), 1366-1387.
  The Morales-Ramis-Simó theory was used and a very useful result for planar systems was obtained and applied.

- Nonintegrability near degenerate equilibria:

$$\mathbf{D}f(\mathbf{0}) = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & -\omega_1 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & \omega_2 & 0 \end{pmatrix}$$

with $\omega_2/\omega_1 \notin \mathbb{Q}$.

Y, Nonintegrability of dynamical systems near degenerate equilibria, *Comm. Math. Phys.*, **398** (2023), 1129–1152.

Y, Nonintegrability of truncated Poincare-Dulac normal forms of resonance degree two, *J. Differential Equations*, **373** (2024), 526–563.

In particular, we use the result to show that the Rösler system and coupled van der Pol oscillators are analytically nonintegrable near equilibria.

- Two-degree-of-freedom Hamiltonian systems:

$$\dot{x} = J\mathrm{D}_x H(x,y), \quad \dot{y} = J\mathrm{D}_y H(x,y), \quad (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Y, Galoisian obstructions to integrability and Melnikov criteria for chaos in two-degree-of-freedom Hamiltonian systems with saddle centres, *Nonlinearity*, **16** (2003), 2003-2012.
Y & Yamanaka, Nonintegrability of dynamical systems with homo- and heteroclinic orbits, *J. Differential Equations*, **263** (2017), 1009–1027.
Y & Yamanaka, Heteroclinic orbits and nonintegrability in two-degree-of-freedom Hamiltonian systems with saddle-centers, *SIGMA*, **15** (2019), 049.

In particular, the following were shown:

- When there exists a saddle-centers, the stable manifolds of periodic orbits near the saddle-center intersect their unstable manifolds transversely if the system is nonintegrable near the manifolds.
- When there exist two saddle-centers,
  the stable manifolds of periodic orbits near one of the saddle-centers may not intersect the unstable manifolds of periodic orbits near the other even if the system is nonintegrable near the manifolds.

- Nonintegrability of semiclassical model by Osawa & Leok (2013) and Osawa (2021):

  Y, Nonintegrability of semiclassical perturbations of single-degree-of-freedom Hamiltonian systems, submitted for publication.

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  Y, Nonintegrability of semiclassical perturbations of single-degree-of-freedom Hamiltonian systems, submitted for publication.

- Integrability of integrable PDEs by quadrature:

  Y, Integrability of the Zakharov-Shabat systems by quadrature, *Comm. Math. Phys.*, **400**, 315–340.