Hyperbolicity for horocyclic perturbations of semi-parabolic Hénon maps

Takumi Yagi

Kyoto university
yagi.takumi.45n@st.kyoto-u.ac.jp

May 29, 2024

Definition

For $(c, a) \in \mathbb{C}^2$ with $a \in \mathbb{D}$, consider the **Hénon map** $H\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$H(x, y) = (x^2 + c + ay, ax)$$

If $a \neq 0$, then H is biholomorphic.

- $H^n := H \circ \cdots \circ H$ (n-times)
- The **filled Julia sets**
  $K^{\pm} := \{(x, y) \in \mathbb{C}^2 : \{H^{\pm n}(x, y)\}_{n \in \mathbb{N}} \textit{ is bounded in } \mathbb{C}^2\}$
- The **Julia sets**
  $J^+ := \partial K^+, \; J^- := \partial K^-, \; J := J^+ \cap J^-$

Definition

A point $\mathbf{q} \in \mathbb{C}^2$ is a **semi-parabolic fixed point** of H if $H(\mathbf{q}) = \mathbf{q}$, the eigenvalues λ, ν of $(DH)_{\mathbf{q}}$ satisfy that λ is a root of unity and $|\nu| < 1$.

Perturbations of semi-parabolic parameters

- $\lambda_0 = \exp(2\pi is/m)$: a primitive root of unity of order m
- $\{\lambda_t\}_{t\in[0,1]}$: a one-parameter continuous family satisfying that

$$\{\lambda_t\}_{t\in(0,1]} \subset \mathbb{D} \text{ or } \{\lambda_t\}_{t\in(0,1]} \subset \mathbb{C} \setminus \overline{\mathbb{D}}$$

- Set $c(a,t) := (1-a^2)\left(\frac{\lambda_t}{2} - \frac{a^2}{2\lambda_t}\right) - \left(\frac{\lambda_t}{2} - \frac{a^2}{2\lambda_t}\right)^2$
- $H_{a,t}(x,y) := (x^2 + c(a,t) + ay, ax)$ has a fixed point with one eigenvalue λ_t.

Then, the family $\{H_{a,t}\}$ is a perturbation of the semi-parabolic Hénon family $\{H_{a,0}\}$.

Definition, McMullen 2000

- Assume that $\mathbb{C} \setminus \{0\} \ni \lambda_t \to \lambda_0$ as $t \to 0$,

$$\lambda_t / \lambda_0 = \exp(L_t + i\theta_t), \ \textit{and} \ \theta_t \to 0.$$

- λ_t/λ_0 converges to 1 **horocyclically** as $t \to 0$ if $\theta_t^2 = o(L_t)$.

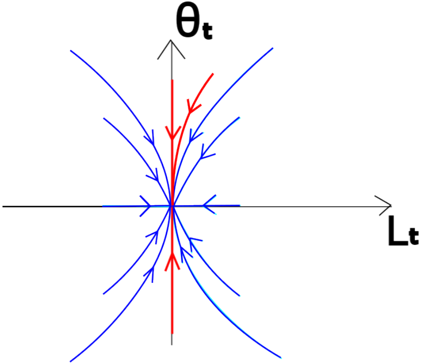


Figure: horocyclic , not horocyclic

Definition

- We say that $\{H_{a,t}\}$ is a **horocyclic perturbation** of $\{H_{a,0}\}$ $\iff \lambda_t/\lambda_0 = \exp(L_t + i\theta_t)$ satisfies $\theta_t^2 = o(L_t)$.

Recall $H_{a,t}(x,y) := (x^2 + c(a,t) + ay, ax)$.

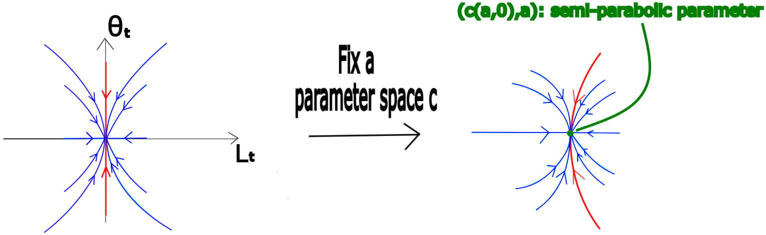


Figure: not horocyclic , horocyclic

Definition

H is **hyperbolic** $\iff \exists$ subbundles E^u, E^s
$\exists N \in \mathbb{N}, K > 1$ s.t. $T\mathbb{C}^2|_J = E^s \oplus E^u, DH(E^l) = E^l \,(l \in \{s, u\})$

$$\|(DH^N)_{\mathbf{p}}(u)\| \geq K\|u\|, \quad \text{and} \quad \|(DH^{-N})_{\mathbf{p}}(s)\| \geq K\|s\|$$

for $\mathbf{p} \in J, \; u \in E^u, s \in E^s$.

- Hénon maps with semi-parabolic fixed points are not hyperbolic.

Theorem, Radu and Tanase 2017 ($\theta_t = 0$)

If $\theta_t = 0$ for $0 \le t \le 1$, $\exists \delta_0$ such that $H_{a,t}$ is hyperbolic for $0 < t < \delta_0$ and $0 < |a| < \delta_0$.

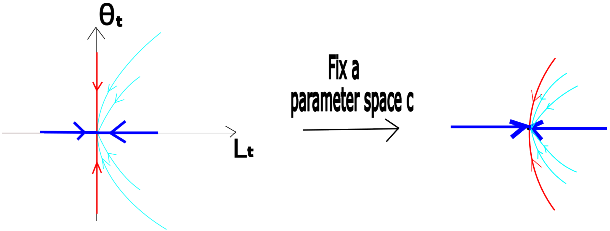


Figure: not horocyclic , horocyclic ($\theta_t = 0$)

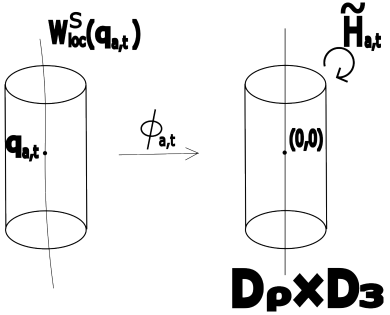
Theorem, (Y)

If $\lambda_0 = 1$, $\theta_t^2 = o(L_t)$ and $L_t > 0$ for $0 < t < \delta_0$, then $H_{a,t}$ is hyperbolic for $0 < |a| < \delta_0$ and $0 < t < \delta_0$.

Lemma, Radu and Tanase 2017 ($\theta_t = 0$)

By a local coordinate $\phi_{a,t}\colon \mathbb{D}_{\rho'}(\mathrm{Pr}_1\mathbf{q}_{a,t}) \times \mathbb{D}_3 \to \mathbb{D}_\rho \times \mathbb{D}_3$,
$(x_1, y_1) = \tilde{H}_{a,t}(x, y) := \phi_{a,t} \circ H_{a,t} \circ \phi_{a,t}^{-1}(x, y)$ satisifes

$$\begin{cases} x_1 = \lambda_t(x + x^2 + b_{a,t,3}(y)x^3 + \cdots) \\ y_1 = \nu_{a,t}y + xh_{a,t}(x, y). \end{cases}$$



We confrim that $\tilde{H}_{a,t}$ is expanding in $\mathbb{D}_\rho \times \mathbb{D}_3$.

Lemma

$\exists$ an open set $V_{a,t} \supset \phi_{a,t}(J_{a,t} \cap (\mathbb{D}_{\rho'}(\mathrm{Pr}_1 \, \mathbf{q}_{a,t} \times \mathbb{D}_3))$,
$\exists \ \beta > 1$ and $\exists$ horizontal cone field $\{\tilde{C}^h_{(x,y)}\}_{(x,y)\in V_{a,t}}$ s.t.

$$(x,y) \in V_{a,t} \quad \Rightarrow \quad \exists N \in \mathbb{N} \text{ s.t. } \|(D\tilde{H}^N_{a,t})_{(x,y)}(\zeta,\eta)\| \geq \beta \|(\zeta,\eta)\|$$
$$\text{for } (\zeta,\eta) \in \tilde{C}^h_{(x,y)}.$$

Neighborhood of the Julia set in $\mathbb{D}_\rho \times \mathbb{D}_3$

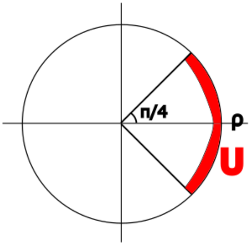
We set

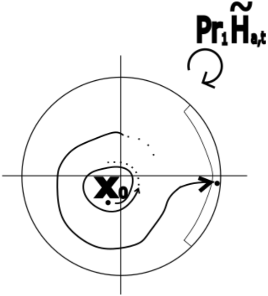
- $U := \{x \in \mathbb{D}_\rho : |\arg x| < \pi/4, |x| > \rho - \rho/10\}$
- $V_{a,t} := \bigcup_{j \geq 0} \tilde{H}_{a,t}^{-j}(U \times \mathbb{D}_3) \cap (\mathbb{D}_3 \times \mathbb{D}_3)$

Lemma

The set $V_{a,t}$ is a neighborhood of $\phi_{a,t}(J_{a,t} \cap (\mathbb{D}_{\rho'}(\mathrm{Pr}_1 \mathbf{q}_{a,t}) \times \mathbb{D}_3))$.

Each point $(x_0, y_0) \in V_{a,t}$ is eventually mapped into $U \times \mathbb{D}_3$ by $\tilde{H}_{a,t}$.





We define the *horizontal cones* $\tilde{C}^h_{(x,y)}$ at $(x, y) \in V_{a,t}$ as

$$\tilde{C}^h_{(x,y)} := \{(\zeta, \eta) \in T_{(x,y)}V_{a,t} : |\zeta| \geq |\eta|\}.$$

Proposition (Invariant)

$(D\tilde{H}_{a,t})_{(x,y)}(\tilde{C}^h_{(x,y)}) \subset \tilde{C}^h_{H(x,y)}$ for $(x, y) \in V_{a,t}$.

For $(\zeta, \eta) \in \tilde{C}^h_{(x,y)}$, we set $\|(\zeta, \eta)\|_{\max} := \max\{|\zeta|, |\eta|\}$.

Computation of derivatives $D\tilde{H}$

We set

- $(x_n, y_n) = \tilde{H}^n_{a,t}(x_0, y_0)$ for $n \in \mathbb{N}$ and $(x_0, y_0) \in V_{a,t}$.
- $(\zeta_0, \eta_0) \in \tilde{C}^h_{(x_0,y_0)}$.

Lemma

$$\begin{aligned} & \|(D\tilde{H}^n_{a,t})_{(x_0,y_0)}(\zeta_0, \eta_0)\|_{\max} \\ \geq \quad & e^{nL_t}\Big(1 + \sum_{j=0}^{n-1} 2|x_j|\cos(\arg x_j) + \sum_{j=0}^{n-1} O(x_j^2)\Big)\|(\zeta_0, \eta_0)\|_{\max} \\ =: \quad & e^{nL_t} K(x_0, y_0, n)\|(\zeta_0, \eta_0)\|_{\max} \end{aligned}$$

Since $L_t > 0$, we show for $(x_0, y_0) \in V_{a,t}$, there exists $N \in \mathbb{N}$ s.t.

$$K(x_0, y_0, N) > 1.$$

Difference between $\theta_t = 0$ and $\theta_t \neq 0$

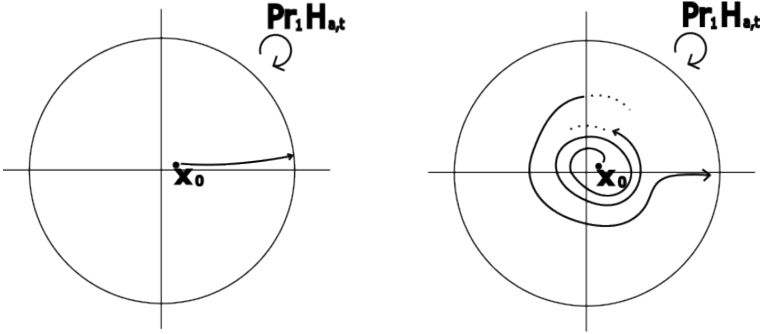


Figure: $\theta_t = 0$ (left), $\theta_t \neq 0$ (right)

Recall $\mathrm{Pr}_1 \tilde{H}_{a,t}(x,y) = \exp(L_t + i\theta_t)(x + x^2 + b_{a,t,3}(y)x^3 + \cdots)$.
If $\theta_t = 0$ for $0 < t < \delta_0$, then

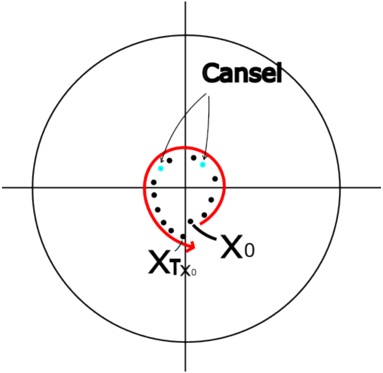
$$K(x_0, y_0, n) = (1 + \sum_{j=0}^{n-1} 2|x_j| \cos(\arg x_j) + \sum_{j=0}^{n-1} O(x_j^2)) > 1.$$

We set $T_{x_0} := \min\{n \in \mathbb{N}_{>2} : |\arg(x_n/x_0)| < 2\theta_t\}$.

Lemma

If $-2\theta_t < \arg\, x_0 + \pi/2 < 2\theta_t$, then

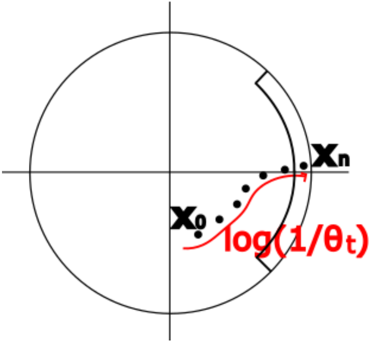
$$K(x_0, y_0, T_{x_0}) = 1 + \sum_{j=0}^{T_{x_0}-1} 2|x_j|\cos(\arg\, x_j) + \sum_{j=0}^{T_{x_0}-1} O(x_j^2) > 1$$



Lemma

- If $\mathrm{Re}\, x_j > 0$ for $0 \le j \le N$ and $x_N \in U$, then

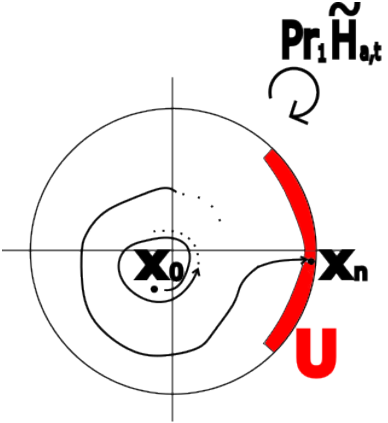
$$K(x_0, y_0, N) = \sum_{j=0}^{N-1} 2|x_j| \cos(\arg\, x_j) + \sum_{j=0}^{N-1} O(x_j^2) \asymp \log(1/|\theta_t|) > 1.$$



Lemma, Expanding on $V_{a,t}$

There is $\beta > 1$ depending on t such that If $(x_0, y_0) \in V_{a,t}$ and $x_n \in U$, then

$$\|(D\tilde{H}^n_{a,t})_{x,y})(\zeta, \eta)\|_{\max} > \beta \|(\zeta, \eta)\|_{\max}.$$



Thank you for your attention!