Theta Operators and Galois Representations for Siegel Modular Forms

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The beginning

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- Representation Theory gives us the set of *Galois representations* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(V)$.
- The Langlands program broadly suggests the following idea

Conjecture (Langlands Program)

Galois representations are in bijection with automorphic forms.

Modular Forms

Definition (Modular Form)

A *modular form of weight k* is a holomorphic function $f : \mathfrak{S}_1 \to \mathbb{C}$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

with Fourier expansion $f(q) = \sum_{n\in\mathbb{Z}_{\geq 0}} a_f(n)q^n$, where $q = e^{2\pi iz}$.

Remark

The above is over $\mathbb{C}$. If the coefficients $a_f(n) \in \overline{\mathbb{F}}_p$, then they are modular forms $\pmod{p}$.

Langlands with modular forms

Theorem (Shimura, Deligne, Serre, Khare-Wintenberger)

Given a normalised eigenform $f \pmod{p}$, there exists a representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ such that:

- $\mathrm{Tr}(\rho_f(\mathrm{Frob}_\ell)) = a_f(\ell)$
- $\det(\rho_f(\mathrm{Frob}_\ell)) = \ell^{k-1}$,

for $\ell \neq p$. Further, if $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is semisimple, continuous, unramified outside p, and odd, then there exists an eigenform f such that $\rho_f \cong \rho$.

An *eigenform* is a simultaneous eigenvector for the Hecke operators $T(n)$. When normalised, the eigenvalue ends up being

$$T(n)f = a_f(n)f.$$

Functoriality

Let's consider an operator on modular forms under the above correspondence.

Definition (Theta Operator)

The *Theta Operator* is the differential operator $\vartheta : M_k \to M_{k+p+1}$,

$$\vartheta = q\frac{d}{dq},$$

which acts on Fourier expansions by

$$(\vartheta f)(\underline{q}) = \sum_{n=0}^{\infty} a_{\vartheta f}(n)q^n = \sum_{n=0}^{\infty} n \cdot a_f(n)q^n.$$

Functoriality

Since under the map ϑ we have that

$$a_f(n) \longmapsto n \cdot a_f(n) \qquad\qquad k \longmapsto k + p + 1,$$

we can determine that

$$\mathrm{Tr}(\rho_f(\mathrm{Frob}_\ell)) = \ell \cdot a_f(\ell), \quad \det(\rho_f(\mathrm{Frob}_\ell)) = \ell^{(k+p+1)-1} \equiv \ell^{k+1},$$

where the last equivalence is due to Fermat's little theorem. From this one can see that

$$\rho_{\vartheta f} = \chi_p \otimes \rho_f$$

where χ_p is the $\pmod{p}$ cyclotomic character, given by

$$\chi_p(\mathrm{Frob}_\ell) = \ell, \quad \text{for} \quad \ell \neq p.$$

A Further Application

With the above, we can find a form of minimal weight giving a representation.

Proposition

Given an eigenform f, there exists an eigenform g of weight $k' \leq p+1$ and an integer $0 \leq i \leq p-1$ such that

$$f(q) = (\vartheta^i g)(q).$$

This, along with the above allows us to "untwist" the Galois representation and relate it to the form of minimal weight.

What next?

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Specifically, since modular forms as above give us 2-dimension representations, we seek higher dimensional analogues.

Siegel Modular Forms

One way to generalise modular forms is the notion of *Siegel modular forms*, which are attached to the symplectic group Sp_{2g} as follows:

Definition (Siegel Modular Form)

A *degree g Siegel modular form of weight k* is a holomorphic function $f : \mathfrak{S}_g \to \mathbb{C}$ such that

$$f\left((AZ+B)(CZ+D)^{-1}\right) = \det(CZ+D)^k f(Z) \quad \text{for } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Some key points

Remark

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Langlands for Siegel Modular Forms

Conjecture

Let f be a degree g Siegel eigenform $\pmod{p}$. *There exists a semisimple continuous group homomorphism*

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GSpin}_{2g+1}(\overline{\mathbb{F}}_p)$$

which is unramified outside p, where $\rho_f(\mathrm{Frob}_\ell) = s_\ell$, the ℓ-Satake parameter of f.

Compare this to the $g = 1$ case. We have the isomorphisms

$$\mathrm{GSpin}_3 \cong \mathrm{GSp}_2 \cong \mathrm{GL}_2,$$

and we recover the previous theorem.

Theta operator

We focus on this generalisation, due to Boecherer-Nagaoka:

Definition (Theta Operator)

The *Theta Operator of Boecherer-Nagaoka* is the differential operator $\vartheta_{BN} : M_k \to M_{k+p+1}$,

$$\vartheta_{BN} = \frac{1}{(2\pi i)^g} \det \begin{pmatrix} \partial/\partial z_{11} & \frac{1}{2}\partial/\partial z_{12} & \cdots & \frac{1}{2}\partial/\partial z_{1g} \\ \frac{1}{2}\partial/\partial z_{12} & \partial/\partial z_{22} & \cdots & \frac{1}{2}\partial/\partial z_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\partial/\partial z_{1g} & \frac{1}{2}\partial/\partial z_{2g} & \cdots & \partial/\partial z_{gg} \end{pmatrix}$$

which acts on Fourier expansions by

$$(\vartheta_{BN} f)(\underline{q}) = \sum_{\underline{n} \in \mathcal{F}(g)} a_{\vartheta_{BN} f}(\underline{n}) \underline{q}^{\underline{n}} = \sum_{\underline{n} \in \mathcal{F}(g)} \det(n) \cdot a_f(\underline{n}) \underline{q}^{\underline{n}}.$$

The Punchline - Functoriality

We have proven the following:

Theorem

Let T be a Hecke operator and let ϑ_{BN} be the theta operator of Boecherer-Nagaoka acting on degree g Siegel modular forms $\pmod{p}$. Then

$$T \circ \vartheta_{BN} = \deg(T) \cdot \vartheta_{BN} \circ T.$$

Specifically,

$$T(n) \circ \vartheta_{BN} = n^g \cdot \vartheta_{BN} \circ T(n).$$

Corollary

Given degree g Siegel eigenform $\pmod{p}$ and a representation $\omega : \mathrm{GSpin}_{2g+1} \to \mathrm{GL}(V)$, then

$$\omega \circ \rho_{\vartheta_{BN} f} = \chi_p^{\otimes g} \otimes (\omega \circ \rho_f)$$

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  - The operator of Flanders-Ghitza, arising from algebraic geometry
- Do the results above lead to a minimal weight statement as in the $g = 1$ case?
- Can we formulate a version of Serre's conjecture for Siegel modular forms? (And prove it, too - I mean, how hard could it be?)

The End

Thanks for your time!