Differentiation of Siegel Modular Forms

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Goal and Motivation

Modular forms are certain holomorphic functions which connect to many other objects in number theory. In particular, they form parts of the Langlands correspondence, and further were and integral part of the proof of Fermat's Last Theorem. Understanding how to manipulate them is an important step in understanding these correspondences.

Noting that the definition includes the word "holomorphic" suggests that there should be a notion of differentiation on modular forms. This turns out to be more subtle than in first appears, and leads to many useful and nontrivial results. Having now been well explored, we are interested in extending this theory to Siegel Modular Forms, a generalisation of modular forms.

Classical Modular Forms

A *modular form of weight* k is a holomorphic function $f : \mathfrak{H}_1 \to \mathbb{C}$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and f is *holomorphic at infinity.* These can be expressed as $f(q) = \sum_{n=0}^{\infty} a(n)q^n$, where $q = e^{2\pi i z}$.

Write $M_k(\mathrm{SL}_2(\mathbb{Z}); \mathbb{C})$ for the space of weight k modular forms. If one considers forms with $a(n) \in R$ for a ring R, then write $M_k(\mathrm{SL}_2(\mathbb{Z}); R)$.

Classical Differentiation

We can apply the differential linear operator $\theta = q\frac{d}{dq}$ to a modular form, and the result will be another series in q, but will it be modular?

In general no, over an arbitrary ring R the resulting series is not modular. However, consider $R = \overline{\mathbb{F}}_p$. Here in fact given $f \in M_k(\mathrm{SL}_2(\mathbb{Z}); \overline{\mathbb{F}}_p)$, then $\theta f \in M_{k+p+1}(\mathrm{SL}_2(\mathbb{Z}); \overline{\mathbb{F}}_p)$ and $(\theta f)(q) = \sum na(n)q^n$.

There are many ways to prove this, the primary two being:

(1) A Rankin-Cohen bracket construction of θ.

(2) An algebraic geometry approach, in which modular forms arise from elliptic curves.

Siegel Modular Forms

Siegel Modular Forms are a multivariate analogue of modular forms.

Let $\kappa : \mathrm{GL}_g(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ be a rational representation. A *Siegel modular form of weight* κ is a holomorphic function $f : \mathfrak{H}_g \to \mathbb{C}^m$ such that

$$f\left((A\mathbf{z}+B)(C\mathbf{z}+D)^{-1}\right) = \kappa(C\mathbf{z}+D)f(\mathbf{z}), \quad \text{where } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Here we have an expansion $f(\mathbf{q}) = \sum_{\mathbf{n}\in\mathcal{F}_g} a(\mathbf{n})\mathbf{q}^{\mathbf{n}}$ and as above can define $M_\kappa(\mathrm{Sp}_{2g}(\mathbb{Z}); R)$ and further $M_k(\mathrm{Sp}_{2g}(\mathbb{Z}); R) = M_{\det(\mathrm{std})^k}(\mathrm{Sp}_{2g}(\mathbb{Z}); R)$.

Geometric Definition

Consider the moduli scheme $\mathcal{A}_g$ of g-dimensional abelian schemes A over R, each with a principal polarization $\lambda : A \xrightarrow{\sim} A^\vee$. Over this there is a universal abelian scheme $Y/\mathcal{A}_g$ which gives us the Hodge bundle $\mathbb{E} = e^*(\Omega_{Y/\mathcal{A}_g})$. Given a rational representation κ, we can "twist" the sheaf $\mathbb{E}$ to produce $\mathbb{E}_\kappa$. We then define

$$M_\kappa(\mathrm{Sp}_{2g}(\mathbb{Z}); R) = H^0(\mathcal{A}_g, \mathbb{E}_\kappa).$$

Differentiation - A first view

The first approach to constructing a differential operator on Siegel Modular Forms mirrors the Rankin-Cohen method above.

Let $f \in M_{k_1}(\mathrm{Sp}_{2g}(\mathbb{Z}); \overline{\mathbb{F}}_p)$, $g \in M_{k_2}(\mathrm{Sp}_{2g}(\mathbb{Z}); \overline{\mathbb{F}}_p)$.
Let $\det(R + xS) = \sum_{i=0}^{g} P_i(R,S)x^i$.
Let

$$Q^{(g)}_{k_1,k_2}(R,S) = \sum_{i=0}^{g} (-1)^i i!(g-i)! \binom{2k_2 - i}{g-i}\binom{2k_1 - g + i}{i} P_i(R,S).$$

If F and G are lifts of f and g to characteristic zero, then

$$[F,G](\mathbf{q}) = Q^{(g)}_{k_1,k_2}(\partial_{\mathbf{q}_1}, \partial_{\mathbf{q}_2})\left(F(\mathbf{q}_1)G(\mathbf{q}_2)\right)|_{\mathbf{q}=\mathbf{q}_1=\mathbf{q}_2}$$

is a Siegel Modular form of weight $k_1 + k_2 + 2$; see [1] Thm 3 or [3] Thm 2.3.

This allows us to define the linear differential operator θ_{BN} by the diagram on the right, where $A(\mathbf{q}) \equiv 1 \pmod p$ is a weight $p-1$ Siegel modular form.

$$\begin{tikzcd} F \arrow[r, "\frac{(-1)^g}{(g+1)!}"] & {[F, A]} \arrow[d, "(\mathrm{mod}\ p)"] \\ f \arrow[u, "\mathrm{lift}"] \arrow[r] \arrow[ur, phantom, "(\mathrm{mod}\ p)"] & \theta_{BN} f \end{tikzcd}$$

Then $\theta_{BN} f \in M_{k_1+p+1}(\mathrm{Sp}_{2g}(\mathbb{Z}); \overline{\mathbb{F}}_p)$ and $(\theta_{BN} f)(\mathbf{q}) = \sum \det(\mathbf{n})a(\mathbf{n})\mathbf{q}^{\mathbf{n}}$.

Differentiation - A second view

One limitation of the above definition is that it only applies for weight $\kappa = \det^k$. Now let us consider $f \in M_\kappa(\mathrm{Sp}_{2g}(\mathbb{Z}); \overline{\mathbb{F}}_p)$.
Turning to the definition of Siegel modular forms where they arise as global sections of a bundle of differential forms, we should expect that there exists a notion of differentiation arising from this. In particular, they are closely related to the de Rham cohomology H^1_{dR}, and thus we can use the *Gauss-Manin connection* ∇. This allows us to define the linear differential operator θ_{FG} by

$$\begin{tikzcd} \mathbb{E}_\kappa \arrow[r] \arrow[drr, bend right, "\theta_{FG}"'] & H^1_{\mathrm{dR}} \arrow[r, "\nabla^\kappa"] & (H^1_{\mathrm{dR}})^\kappa \otimes \Omega^1 \arrow[d, "\mathrm{id} \otimes \mathrm{KS}^{-1} \otimes h"] \\ & & (\mathbb{E} \oplus \mathbb{E}^\vee)^\kappa \otimes \mathrm{Sym}^2\mathbb{E} \otimes \underline{\omega}^{\otimes(p-1)} \arrow[d] \\ & & \mathbb{E}_\kappa \otimes \underline{\omega}^{\otimes(p-1)} \otimes \mathrm{Sym}^2\mathbb{E} \end{tikzcd}$$

So in this case we get that $\theta_{FG} f \in M_{\kappa\otimes\det^{\otimes p-1}\otimes\mathrm{Sym}^2}(\mathrm{Sp}_{2g}(\mathbb{Z}); \overline{\mathbb{F}}_p)$ and $(\theta_{FG} f)(\mathbf{q}) = \sum \mathbf{n} \otimes a(\mathbf{n})\mathbf{q}^{\mathbf{n}}$, see [4].

An application - Galois Representations

One of the main reasons to care about these operators is that they have a clear arithmetic effect on the $\mathbf{q}$-expansions. This has various number-theoretic implications, not least of which is how they affect Galois representations. If one takes the conjectural Galois representation ρ_f attached to a Siegel modular form f and composes with a representation ω_λ of highest weight λ, then

$$\omega_\lambda \circ \rho_{\theta_{FG} f} = \chi^{\alpha(\lambda)} \otimes (\omega_\lambda \circ \rho_f),$$

and

$$\omega_\lambda \circ \rho_{\theta_{BN} f} = \chi^{g\cdot\alpha(\lambda)} \otimes (\omega_\lambda \circ \rho_f).$$

For further details on the above, see [5].

References

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