Theta Operators on Hecke Eigenvalues

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Overview

Modular Forms

Definition (Modular Form)

A *modular form of weight k and level N* is a holomorphic function $f : \mathbb{H}_1 \to \mathbb{C}$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N),$$

where

$$\Gamma(N) = \ker\left(\ \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \ \right),$$

and f is *holomorphic at the cusps*.

The Basic Theory

In fact, modular forms are extremely explicit and nice to study. For example, when $N = 1$ we have the following classification.

Example

$$M_*(\mathrm{SL}_2(\mathbb{Z}); \mathbb{C}) = \mathbb{C}[E_4, E_6],$$

where

$$E_k(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \; q = e^{2\pi i z}.$$

We can also construct spaces $M_k(\Gamma(N); \overline{\mathbb{F}}_p)$ by tensoring (reducing mod p).

Siegel Modular Forms

Definition (Siegel Modular Form)

Let $\kappa : \mathrm{GL}_g(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ be a rational representation. A *Siegel modular form of degree g, weight κ and level N* is a holomorphic function $F : \mathbb{H}_g \to \mathbb{C}^m$ such that

$$F((A\mathbf{z}+B)(C\mathbf{z}+D)^{-1}) = \kappa(C\mathbf{z}+D)F(\mathbf{z}), \quad \text{for} \quad \gamma \in \Gamma^g(N),$$

where

$$\Gamma^g(N) = \ker\left(\ \mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})\ \right)$$

<u>Note:</u> When the weight $\kappa = \det^k$, we will just write k.

The "Basic" Theory

Here things are more tricky. There are quite a few obstacles in even computing examples. There are still some nice qualities, like the following when $\kappa = \det^k$ and $g = 2$, $N = 1$.

Theorem (Igusa Generators)

$$M_*(\mathrm{Sp}_4(\mathbb{Z}); \mathbb{C}) = \mathbb{C}[\phi_4, \phi_6, \chi_{10}, \chi_{12}, \chi_{35}]/R$$

Note that we have **q**-expansions of the form

$$f(\mathbf{q}_N) = \sum_{\mathbf{n}} \mathbf{a}(\mathbf{n})\mathbf{q}_N^{\mathbf{n}}, \mathbf{q}_N^{\mathbf{n}} = e^{\frac{2\pi i}{N}\mathrm{Tr}(\mathbf{nz})}.$$

The Slash Operator

Definition (Weight κ slash operator)

The *weight κ slash operator* $|_\kappa$ gives an action of $\mathrm{GSp}_{2g}(\mathbb{Q})^+$ on weight κ forms by

$$(f|_\kappa)(\mathbf{z}) = \eta(\gamma)^{\sum \lambda_i - g(g+1)/2} \kappa(C\mathbf{z} + D)^{-1} f(\gamma \mathbf{z}),$$

where $(\lambda_i \geq \ldots \geq \lambda_g)$ is the highest weight of κ.

Note that if $\gamma \in \Gamma^g(N)$, then $(f|_\kappa)(\mathbf{z}) = f(\mathbf{z})$.

Hecke Operators

The slash operator is well-defined on cosets $\Gamma^g(N)\gamma$. Given a double coset $\Gamma^g(N)\gamma\Gamma^g(N)$, it can be expressed as a union of cosets as

$$\Gamma^g(N)\gamma\Gamma^g(N) = \bigcup_j \Gamma^g(N)\gamma_j.$$

Definition (Hecke Operator)

Let $\gamma \in \mathrm{GSp}_{2g}(\mathbb{Q})^+$ have a coset expansion as above. The *Hecke operator* T_γ is

$$(T_\gamma f)(\mathbf{z}) = \sum_j (f|_\kappa \gamma_j)(\mathbf{z}).$$

Eigenforms and Eigenvalues

Definition (Eigenform)

An *eigenform* is a form f which is an simultaneous eigenvector for all the Hecke operators T, i.e. $\exists \Psi_f(T) \in \overline{\mathbb{F}}_p$ such that

$$Tf = \Psi_f(T) f.$$

Definition (Eigensystem)

Let $\mathcal{H}$ be the algebra of Hecke operators and f an eigenform. The function

$$\begin{aligned} \Psi_f : \quad \mathcal{H} &\longrightarrow \overline{\mathbb{F}}_p \\ T &\longmapsto \Psi_f(T) \end{aligned}$$

is called the *Hecke eigensystem* of f.

The Theta Operator

The word *holomorphic* may lead one to think that modular forms should be differentiable. This turns out to be true (mod p), and for $g = 1$ we have the operator

$$\begin{array}{rccc} \vartheta : & M_k(\Gamma(N);\overline{\mathbb{F}}_p) & \longrightarrow & M_{k+p+1}(\Gamma(N);\overline{\mathbb{F}}_p) \\ & f & \longmapsto & \frac{1}{2\pi i}\frac{d}{dz}f(z) = \frac{d}{dq}f(q). \end{array}$$

One can construct this many ways, either using Rankin-Cohen brackets or algebraic geometry.

Siegel Theta Operators

We'd like a similar differential operator on Siegel modular forms.

As it turns out, there are many generalisations of ϑ.

- ϑ_{BN} from Boecherer-Nagaoka, using Rankin-Cohen-type brackets.
- ϑ_{FG} from Flander-Ghitza, using algebraic geometry.
- Various operators from Yamauchi, using algebraic geometry in the special case $g = 2$.

Boecherer-Nagaoka

A theorem of Eholzer-Ibukiyama gives us a differential bracket taking two forms F, G of weight k_1, k_2 in characteristic 0 to a form $[F, G]$ of weight $k_1 + k_2 + 2$.

There is a form called the *Hasse invariant* H which reduced (mod p) is the constant 1. So one defines

$$\vartheta^0_{BN} f = [F, H]$$

and then splitting $\vartheta^0_{BN} = \vartheta^1_{BN} + p\vartheta^2_{BN}$, one defines

$$\vartheta_{BN} = \text{reduction of } \frac{(-1)^g}{(g+1)!}\vartheta^1_{BN} \pmod{p}.$$

Boecherer-Nagaoka

Theorem

The operator ϑ_{BN} acts on Siegel modular forms by

$$\begin{array}{rccc} \vartheta_{BN}: & M_k(\Gamma^g(N);\overline{\mathbb{F}}_p) & \longrightarrow & M_{k+p+1}(\Gamma^g(N);\overline{\mathbb{F}}_p) \\ & f(\mathbf{q}) & \longmapsto & \det(\partial_{\mathbf{q}})f(\mathbf{q}), \end{array}$$

i.e.

$$(\vartheta_{BN}f)(\mathbf{q}) = \frac{1}{N^g}\sum_{\mathbf{n}} \det(\mathbf{n})a(\mathbf{n})\mathbf{q}_N^{\mathbf{n}}.$$

Flander-Ghitza

Simplified construction:

$$
\begin{aligned}
\vartheta_{FG} : \mathbb{E}_\kappa \hookrightarrow (H^1_{\mathrm{dR}})^\lambda
&\xrightarrow{\nabla^\lambda} (H^1_{\mathrm{dR}}) \otimes \Omega^1 \\
&\xrightarrow{\mathrm{id}\otimes KS^{-1}\otimes h} (\mathbb{E} \oplus \mathbb{E}^\vee)^\lambda \otimes \mathrm{Sym}^2\,\mathbb{E} \otimes \omega^{\otimes(p-1)} \\
&\longrightarrow \mathbb{E}_\kappa \otimes \underline{\omega}^{(p-1)} \otimes \mathrm{Sym}^2\,\mathbb{E}.
\end{aligned}
$$

Important features:

- $\mathbb{E}$ is the Hodge bundle on $\mathcal{A}_{g,N}$.
- ∇ is the Gauss-Manin connection.
- KS is the Kodaira-Spencer isomorphism.

Flander-Ghitza

Theorem

The operator ϑ_{FG} acts on Siegel modular forms by

$$\begin{array}{rccc} \vartheta_{FG}: & M_\kappa(\Gamma^g(N);\overline{\mathbb{F}}_p) & \longrightarrow & M_{\kappa\otimes\det^{(p-1)}\otimes\mathrm{Sym}^2}(\Gamma^g(N);\overline{\mathbb{F}}_p) \\ & f(\mathbf{q}) & \longmapsto & (\vartheta_{FG}f)(\mathbf{q}), \end{array}$$

where

$$(\vartheta_{FG}f)(\mathbf{q}) = \sum_{\mathbf{n}} (\mathbf{n}\otimes\mathbf{a}(\mathbf{n}))\mathbf{q}_N^{\mathbf{n}}.$$

Yamauchi

The operator of Flander-Ghitza ends in a projection to make the weight irreducible. Working with $g = 2$, Yamauchi does this in a highly explicit way, achieveing multiple maps ϑ_i. Further, if one begins with $\kappa = \det^k$ and applies ϑ twice, one gets a new map Θ from k to $k + p + 1$, which is in fact the same as ϑ_{BN}. We will thus discuss these operators in no further detail.

Thus in $g = 1$ and $g = 2$ we have a direct relationship between the Rankin-Cohen bracket and algebro-geometric constructions.

Theta on Eigenvalues

One of the most fascinating properties of the theta operator on modular forms its the following.

Theorem

Let f be an eigenform with eigensystem Ψ_f. If $\vartheta f \neq 0$, then ϑf is an eigenform and

$$\Psi_{\vartheta f}(T_\gamma) = \det(\gamma)\Psi_f(T_\gamma).$$

One proves this by demonstrating the commutation relation

$$T_\gamma \circ \vartheta = \det(\gamma) \cdot \vartheta \circ T_\gamma$$

Main Theorem

Theorem

Let f be an eigenform with eigensystem Ψ_f. Then

(1) *If $\vartheta_{BN} f \neq 0$, then $\vartheta_{BN} f$ is an eigenform and*

$$\Psi_{\vartheta_{BN} f}(T_\gamma) = \det(\gamma)\Psi_f(T_\gamma).$$

(2) *If $\vartheta_{FG} f \neq 0$, then $\vartheta_{FG} f$ is an eigenform and*

$$\Psi_{\vartheta_{FG} f}(T_\gamma) = \eta(\gamma)\Psi_f(T_\gamma).$$

We will describe the proof for part (1).

Some lemmas

Lemma

A double coset $\Gamma^g(N)\gamma\Gamma^g(N)$ can be decomposed into right cosets of the form

$$\Gamma^g(N)\begin{pmatrix}\eta(\gamma)(D^\top)^{-1} & B \\ 0 & D\end{pmatrix}.$$

Lemma

If $\gamma = \begin{pmatrix}\eta(\gamma)(D^\top)^{-1} & B \\ 0 & D\end{pmatrix}$ *then*

$$(|_{k+p+1}\gamma)\circ\vartheta^1_{BN} = \det(\gamma)\cdot\vartheta^1_{BN}\circ(|_k\gamma).$$

The advantage of having a block upper triangular matrix is that we can do explicit calculations on **q**-expansions.

Furthermore, splitting $\vartheta^0_{BN} = \vartheta^1_{BN} + p\vartheta^2_{BN}$ allows us to avoid certain issues when coming to reduction (mod p).

Proof of second Lemma.

We will omit all the actual effort and just show the big impressive equations.

$$
\begin{aligned}
((\vartheta^1_{BN}F)|_{k+p+1}\gamma)(\mathbf{q}) = \left(\eta(\gamma)^g \det(D)^{-1}\right)^{p-1} \eta(\gamma)^g \\
\times \frac{1}{N^g}\eta(\gamma)^{(k+1)g-g(g+1)/2}\det(D)^{-(k+2)} \\
\times \sum_{\mathbf{n}} \det(\mathbf{n})a(\mathbf{n})c(\mathbf{n})\mathbf{q}_N^{\mathbf{n}'},
\end{aligned}
$$

$$
\begin{aligned}
(\vartheta^1_{BN}(F|_k))(\mathbf{q}) = \frac{1}{N^g}\eta(\gamma)^{(k+1)g-g(g+1)/2}\det(D)^{-(k+2)} \\
\times \sum_{\mathbf{n}} \det(\mathbf{n})a(\mathbf{n})c(\mathbf{n})q_N^{\mathbf{n}'},
\end{aligned}
$$

where $\mathbf{n}' = \eta(\gamma)D^{-1}\mathbf{n}(D^\top)^{-1}$ and $c(\mathbf{n}) = e^{2\pi i \operatorname{Tr}(\mathbf{n}BD^{-1})/N}$. □

Proof of Theorem part (1).

By the first lemma, we can expand

$$\Gamma^g(N)\gamma\Gamma^g(N) = \bigcup_j \Gamma^g(N)\gamma_j$$

where γ_j is block upper triangular, and $\det(\gamma_j) = \det(\gamma)$.

By the second lemma, these each commute with the slash operator, up to $\det(\gamma)$. Thus, the result follows. $\square$

Some Further Questions

1. Relationship between theta constructions
2. Generalised Rankin-Cohen brackets
3. Theta cycles
4. "Weight in Serre's Conjecture" analogue

Thanks for listening!