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Today's topics

1 Average to instantaneous velocity, geometric idea

Briggs–Cochran–Gillett §2.1 pp. 54–60

Last time, we looked at the average velocity over various intervals $[0, h]$ (with $h > 0$) of an object whose position is given by a function f, and used this to make a conjecture about the instantaneous velocity at 0. Now let's look at a more geometric perspective on the same phenomenon.

Example 1 (§2.1 **Ex. 28)** *Let $f(x) = x^3 - x$. Make a table of slopes of secant lines and make a conjecture about the slope of the tangent line at $x = 1$. (Recall that a **secant line** is a straight line joining two points on a curve.)*

For the interval $[1, t]$, the slope of the secant line is

$$\frac{(t^3 - t) - (1^3 - 1)}{t - 1} = \frac{t^3 - t}{t - 1}.$$

Here's one possible table (you could choose other endpoints):

Interval	Slope of secant line
$[1, 2]$	6
$[1, 1.5]$	3.75
$[1, 1.1]$	2.31
$[1, 1.01]$	2.0301
$[1, 1.001]$	2.003001

This strongly suggests that the slope of the tangent line at $x = 1$ is 2.

To summarize:

The instantaneous velocity at $t = t_0$	=	"Limit as $t \to t_0$ of the average velocities in the intervals $[t_0, t]$"

This geometrically corresponds to the following:

The slope of the tangent line to $s(t)$ at $(t_0, s(t_0))$	=	"Limit as $t \to t_0$ of the slopes of the secant lines between $(t_0, s(t_0))$ and $(t, s(t))$"

2 Definition of limits

Briggs-Cochran-Gillett § 2.2, pp. 61–68

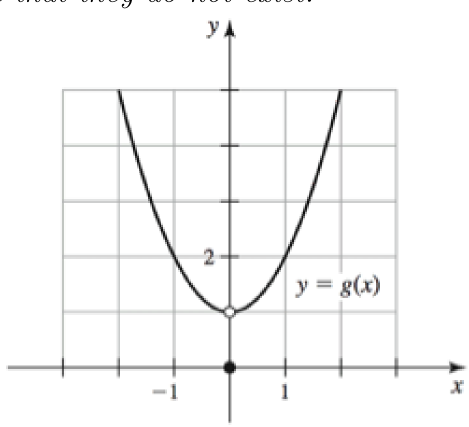
Definition 2 (Limit of a function (Preliminary)) *Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write*

$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

2.1 Finding limits with graphs

Example 3 (§2.2 Ex. 8) *Use the graph of g in the figure to find the following values or state that they do not exist.*



(a) $g(0)$

(b) $\lim_{x \to 0} g(x)$

(c) $g(1)$

(d) $\lim_{x \to 1} g(x)$

2.2 Finding limits with tables

Example 4 (§2.2 Ex. 12) *Let* $f(x) = \frac{x^3-1}{x-1}$.

(a) Calculate $f(x)$ *for each value of* x *in the following table.*

x	0.9	0.99	0.999	0.9999
$f(x) = \dfrac{x^3-1}{x-1}$				
x	1.1	1.01	1.001	1.0001
$f(x) = \dfrac{x^3-1}{x-1}$				

(b) Make a conjecture about the value of $\lim_{x \to 1} \dfrac{x^3-1}{x-1}$.

2.3 One-sided limits

Definition 5 (One-sided Limits: a right-sided limit or a left-sided limit)
Right-sided limit*: Suppose* f *is defined for all* x *near* a *with* $x > a$*. If* $f(x)$ *is arbitrarily close to* L *for all* x *sufficiently close to* a *with* $x > a$*, we write*

$$\lim_{x \to a^+} f(x) = L$$

and say the limit of $f(x)$ *as* x *approaches* a *from the right equals* L*.*
Left-sided limit*: Suppose* f *is defined for all* x *near* a *with* $x < a$*. If* $f(x)$ *is arbitrarily close to* L *for all* x *sufficiently close to* a *with* $x < a$*, we write*

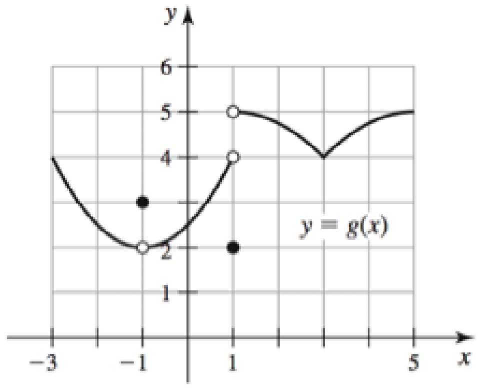
$$\lim_{x \to a^-} f(x) = L$$

and say that the limit of $f(x)$ *as* x *approaches* a *from the left equals* L*.*

Note (one-sided versus two-sided limits): The limit $\lim_{x \to a} f(x) = L$ is a *two-sided limit* because $f(x)$ approaches L as x approaches a for values of x less than a *and* for values of x greater than a.

Theorem 6 (Relationship between one-sided and two-sided limits) *Assume* f *is defined for all* x *near* a *except possibly at* a*. Then* $\lim_{x \to a} f(x) = L$ *if and only if* $\lim_{x \to a^+} f(x) = L$ *and* $\lim_{x \to a^-} f(x) = L$*.*

Example 7 (§2.2 Ex. 24) *Use the graph of g in the figure to find the following values or state that they do not exist. If a limit does not exist, explain why.*



(a) $g(-1)$

(b) $\lim_{x\to -1^-} g(x)$

(c) $\lim_{x\to -1^+} g(x)$

(d) $\lim_{x\to -1} g(x)$

(e) $g(1)$

(f) $\lim_{x\to 1} g(x)$

(g) $\lim_{x\to 3} g(x)$

(h) $g(5)$

(i) $\lim_{x\to 5^-} g(x)$