Dr. Daniel Hast, *drhast@bu.edu*

Today's topics

1 Techniques for computing limits

1.1 One-sided limits

Theorem 1 (One-sided limit laws) *Limit laws 1-6 (sum, difference, constant multiple, product, quotient, and power) hold with $\lim_{x\to a}$ replaced with $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$. Law 7 is modified as follows. Assume $n > 0$ is an integer.*

7. ***Root***

(a) $\lim_{x\to a^+}(f(x))^{1/n} = \left(\lim_{x\to a^+} f(x)\right)^{1/n}$*, provided $f(x) \geq 0$ for all x near a with $x > a$ if n is even.*

(b) $\lim_{x\to a^-}(f(x))^{1/n} = \left(\lim_{x\to a^-} f(x)\right)^{1/n}$*, provided $f(x) \geq 0$ for all x near a with $x < a$ if n is even.*

Example 2 *Let*

$$f(x) = \begin{cases} 0 & \text{if } x \leq -5 \\ \sqrt{25 - x^2} & \text{if } -5 < x < 5 \\ 3x & \text{if } x \geq 5. \end{cases}$$

Compute the following limits or state that they do not exist.

(a) $\lim_{x\to -5^-} f(x)$

(b) $\lim_{x\to -5^+} f(x)$

(c) $\lim_{x\to -5} f(x)$

(d) $\lim_{x\to 5^-} f(x)$

(e) $\lim_{x\to 5^+} f(x)$

(f) $\lim_{x\to 5} f(x)$

Example 3 *Compute the following limits or state that they do not exist.*

(a) $\lim_{x\to 4} \frac{x^2 - 2x - 3}{x - 3}$

(b) $\lim_{x\to 3} \frac{x^2 - 2x - 3}{x - 3}$

Example 4 *Evaluate* $\lim_{x\to a} \frac{x^2 - a^2}{\sqrt{x} - \sqrt{a}}$, *where* $a > 0$ *is a real number.*

1.2 Other techniques

Example 5 *Here we analyze the behavior of* $x \sin \frac{1}{x}$ *near* $x = 0$*:*

1. *Show that* $-|x| \leq x \sin \frac{1}{x} \leq |x|$ *for* $x \neq 0$.
2. *Illustrate the inequalities above with a graph.*
3. *Make a conjecture about* $\lim_{x\to 0} x \sin \frac{1}{x}$.

Theorem 6 (Squeeze Theorem) *Assume functions* f, g, h *satisfy* $f(x) \leq g(x) \leq h(x)$ *for all values of* x *near* a*, except possibly at* a*. If*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

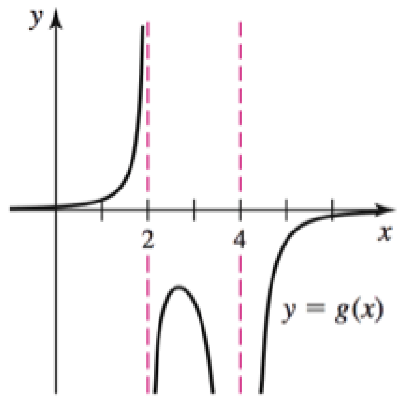
then

$$\lim_{x\to a} g(x) = L.$$

1.3 Infinite limits and one-sided infinite limits

We will now look at functions for which *values increase or decrease without a bound near a point.*

Example 7 *Consider the graph of* g *in the figure. Analyze the following limits.*



1. $\lim_{x \to 2^-} g(x)$
2. $\lim_{x \to 2^+} g(x)$
3. $\lim_{x \to 2} g(x)$
4. $\lim_{x \to 4^-} g(x)$
5. $\lim_{x \to 4^+} g(x)$
6. $\lim_{x \to 4} g(x)$

Definition 8 (Infinite Limits and One-sided Limits)
Suppose f is defined for all x near a.

- *If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a we write*

$$\lim_{x \to a} f(x) = \infty$$

*and we say that the limit of $f(x)$ as x approaches **is infinity**.*

- *If $f(x)$ is negative and grows arbitrarily large in magnitude for all x sufficiently close (but not equal) to a we write*

$$\lim_{x \to a} f(x) = -\infty$$

*and we say that the limit of $f(x)$ as x approaches a **is negative infinity**.*

*In both cases, the limit **does not exist**.*
Suppose now that f is defined for all x near a with $x > a$. If $f(x)$ grows arbitrarily large for all x sufficiently close (but not equal) to a with $x > a$ we write

$$\lim_{x \to a^+} f(x) = \infty.$$

The one sided limits $\lim_{x \to a^+} f(x) = -\infty$, $\lim_{x \to a^-} f(x) = \infty$ and $\lim_{x \to a^-} f(x) = -\infty$ are defined analogously.

1.3.1 Finding infinite limits analytically

Example 9 *Determine the following limits or state that they do not exist.*

1. $\lim_{x \to -2^+} \dfrac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$
2. $\lim_{x \to -2^-} \dfrac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$

3. $\displaystyle\lim_{x\to -2} \frac{x^3-5x^2+6x}{x^4-4x^2}$

4. $\displaystyle\lim_{x\to 2} \frac{x^3-5x^2+6x}{x^4-4x^2}$

1.4 Vertical asymptotes

Definition 10 (Vertical asymptotes)
If $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^+} f(x) = \pm\infty$, *or* $\lim_{x\to a^-} f(x) = \pm\infty$ *then the line* $x = a$ *is called a* ***vertical asymptote****.*

Example 11 *Find all vertical asymptotes* $x = a$ *of the following functions. For each value of* a *determine* $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to a} f(x)$.

1. $f(x) = \dfrac{\cos x}{x^2+2x}$

2. $f(x) = \dfrac{x+1}{x^3-4x^2+4x}$