Dr. Daniel Hast, *drhast@bu.edu*

Today's topics

1 Infinite limits and vertical asymptotes, continued

Briggs–Cochran–Gillett §2.4, pp. 83–91.

Example 1. Determine the following limits or state that they do not exist.

1. $\displaystyle\lim_{x\to -2^+} \frac{x^3-5x^2+6x}{x^4-4x^2}$
2. $\displaystyle\lim_{x\to -2^-} \frac{x^3-5x^2+6x}{x^4-4x^2}$
3. $\displaystyle\lim_{x\to -2} \frac{x^3-5x^2+6x}{x^4-4x^2}$
4. $\displaystyle\lim_{x\to 2} \frac{x^3-5x^2+6x}{x^4-4x^2}$

Example 2. Find all vertical asymptotes $x = a$ of the following functions. For each value of a determine $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a} f(x)$.

1. $f(x) = \dfrac{\cos x}{x^2+2x}$
2. $f(x) = \dfrac{x+1}{x^3-4x^2+4x}$

2 Limits at infinity

Briggs-Cochran-Gillett §2.5, pp. 91–102.

2.1 Limits at infinity and horizontal asymptotes

Definition 3 (Limits at infinity and horizontal asymptotes)**.** *If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write*

$$\lim_{x\to\infty} f(x) = L,$$

and we say that the ***limit of*** $f(x)$ ***as*** x ***approaches infinity is L****.*

In this case the line $y = L$ is a ***horizontal asymptote of*** f*.*

The limit at negative infinity $\lim_{x\to-\infty} f(x) = M$ is defined analogously. When it exists, $y = M$ is also called a horizontal asymptote.

Example 4. Evaluate

$$\lim_{x\to\infty}\left(5 + \frac{1}{x} + \frac{10}{x^2}\right)$$

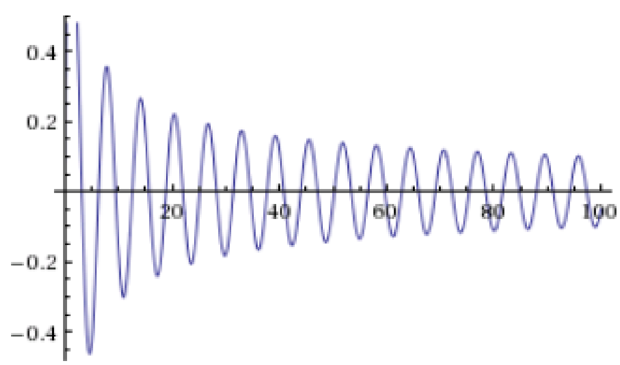
Example 5. Consider the function

$$f(x) = \frac{4x^2 - 7}{8x^2 + 5x + 2}.$$

Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Then give the horizontal asymptotes of f (if any).

Remark 6. *Note that the graph of f* ***can*** *intersect its horizontal asymptote!*

Example 7. Consider the function $f(x) = \dfrac{\sin x}{\sqrt{x}}$:



What is the horizontal asymptote here?

2.2 Infinite limits at infinity

Definition 8 (Infinite limits at infinity)**.** *If f becomes arbitrarily large as x becomes arbitrarily large, then we write $\lim_{x\to\infty} f(x) = +\infty$. The limits $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ are defined analogously.*

Example 9. Determine the limit $\lim_{x\to-\infty}(2x^{-8} + 4x^3)$.

2.3 End behavior

Theorem 10 (End behavior of functions)**.** *Let n, m be a positive integers and $p(x) = a_m x^m + \ldots + a_1 x + a_0$, $q(x) = b_n x^n + \ldots + b_1 x + b_0$ polynomials with $a_m, b_n \neq 0$.*

1. $\lim_{x\to\pm\infty} x^n = \infty$ *when n is even;*
2. $\lim_{x\to\infty} x^n = \infty$ *and* $\lim_{x\to-\infty} x^n = -\infty$ *when n is odd;*
3. $\lim_{x\to\infty} x^{-n} = \lim_{x\to\infty} \frac{1}{x^n} = 0$;
4. $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_m x^m$;
5. *If $m < n$ (degree of numerator less than that of denominator) then* $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = 0$;
6. *If $m = n$ (degree of numerator equal to that of denominator) then* $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \frac{a_m}{b_n}$;
7. *If $m > n$ (degree of numerator greater than that of denominator) then* $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \infty$ *or* $-\infty$;
8. $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to-\infty} e^x = 0$, $\lim_{x\to\infty} e^{-x} = 0$, $\lim_{x\to-\infty} e^{-x} = \infty$;
9. $\lim_{x\to 0^+} \ln(x) = -\infty$, $\lim_{x\to\infty} \ln(x) = \infty$;