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Today's topics

1 Limits at infinity

1.1 End behavior

Theorem 1 (End behavior of functions). *Let n, m be a positive integers and $p(x) = a_m x^m + \ldots + a_1 x + a_0$, $q(x) = b_n x^n + \ldots + b_1 x + b_0$ polynomials with $a_m, b_n \neq 0$.*

1. $\lim_{x\to\pm\infty} x^n = \infty$ *when n is even;*
2. $\lim_{x\to\infty} x^n = \infty$ *and* $\lim_{x\to-\infty} x^n = -\infty$ *when n is odd;*
3. $\lim_{x\to\infty} x^{-n} = \lim_{x\to\infty} \frac{1}{x^n} = 0$*;*
4. $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_m x^m$*;*
5. *If $m < n$ (degree of numerator less than that of denominator) then* $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = 0$*;*
6. *If $m = n$ (degree of numerator equal to that of denominator) then* $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \frac{a_m}{b_n}$*;*
7. *If $m > n$ (degree of numerator greater than that of denominator) then* $\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \infty$ *or* $-\infty$*;*
8. $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to-\infty} e^x = 0$, $\lim_{x\to\infty} e^{-x} = 0$, $\lim_{x\to-\infty} e^{-x} = \infty$*;*
9. $\lim_{x\to 0^+} \ln(x) = -\infty$, $\lim_{x\to\infty} \ln(x) = \infty$*;*

Example 2. Determine the given limits and the give the horizontal asymptotes of f (if any).

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} = \lim_{x\to\infty} \frac{x^{-1}\sqrt{x^2+1}}{x^{-1}(2x+1)} = \lim_{x\to\infty} \frac{\sqrt{x^{-2}}\sqrt{x^2+1}}{x^{-1}(2x+1)} = \lim_{x\to\infty} \frac{\sqrt{1+x^{-2}}}{2+x^{-1}} = \frac{\sqrt{1+0}}{2+0} = \frac{1}{2},$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x+1} = \lim_{x\to-\infty} \frac{x^{-1}\sqrt{x^2+1}}{x^{-1}(2x+1)} = \lim_{x\to-\infty} \frac{-\sqrt{x^{-2}}\sqrt{x^2+1}}{x^{-1}(2x+1)} = \lim_{x\to-\infty} \frac{-\sqrt{1+x^{-2}}}{2+x^{-1}} = -\frac{1}{2}.$$

(Make sure you understand why the second limit is $-1/2$, not $1/2$; it's easy to make a mistake and get the sign wrong there.)

2 Continuity

Briggs–Cochran–Gillett §2.6, pp. 103–115.

Informally, a function f is *continuous* at a if the graph of f does not have a hole or break at a (that is, if the graph near a can be drawn without lifting the pencil). If a function is not continuous at a, then a is a point of discontinuity. We have already seen a number of functions that are continuous:

Theorem 3. *Polynomial functions are continuous for all x. A rational function (a function of the form p/q where p, q are polynomials) is continuous for all x for which $q(x) \neq 0$.*

The informal description of continuity above is sufficient for determining the continuity of simple functions, but it is not precise enough to deal with more complicated functions.

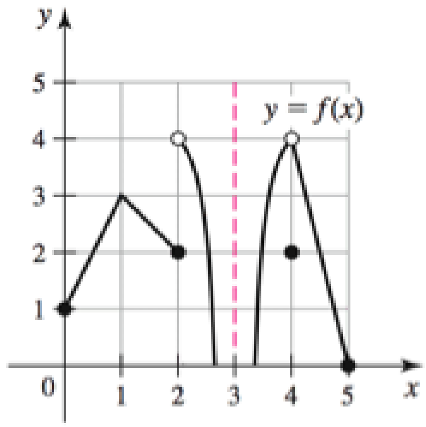
Definition 4 (Continuity at a point)**.** *A function f is* continuous *at a if $\lim_{x\to a} f(x) = f(a)$. If f is not continuous at a, then a is a point of discontinuity.*

2.1 Continuity checklist

In order for f to be continuous at a as in the definition above, the following three conditions must hold:

1. $f(a)$ is defined (a is in the domain of f).
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$ (the value of f equals the limit of f at a).

Example 5. Determine the points at which f has discontinuities.



We have some helpful results about composite functions:

Theorem 6 (Continuity of composite functions at a point)**.** *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.*

Theorem 7 (Limits of composite functions)**.** *If $\lim_{x \to a} g(x) = L$ and f is continuous at L, then*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

2.2 Continuity on an interval

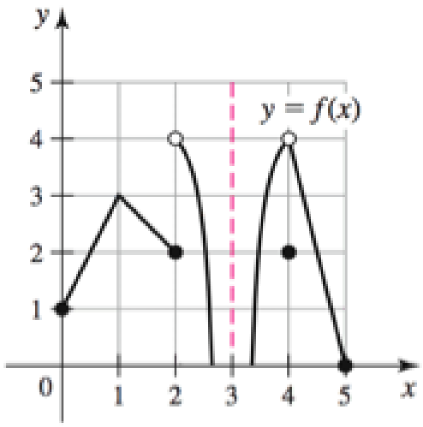
When we talk about continuity of functions, to be more precise, we ought to specify the interval of the function we are considering. A function is *continuous on an interval* if it is continuous at every point in that interval. We must be careful about endpoints of intervals:

Definition 8 (Continuity at endpoints)**.** *A function f is continuous from the left (or left-continuous) at a is $\lim_{x \to a^-} f(x) = f(a)$, and f is continuous from the right (or right-continuous) at a if $\lim_{x \to a^+} f(x) = f(a)$.*

Combining the definitions of left-continuous and right continuous with the definition of continuity at a point (Definition 4), we define what it means for a function to be continuous on an interval:

Definition 9 (Continuity on an interval)**.** *A function f is continuous on an interval I if it is continuous at all points of I. If I contains its endpoints, continuity on I means continuous from the right or left at the endpoints.*

Example 10. Give the intervals where f is continuous.



Example 11. Determine whether f is continuous at $a = 3$.

$$f(x) = \begin{cases} \frac{x^2-4x+3}{x-3} & \text{if} \quad x \neq 3 \\ 2 & \text{if} \quad x = 3 \end{cases}$$

Example 12. Let

$$f(x) = \begin{cases} 2x & \text{if} \quad x < 1 \\ x^2 + 3x & \text{if} \quad x \geq 1 \end{cases}$$

1. Is f continuous at 1?
2. Is f continuous from the left or right at 1?
3. State the interval(s) of continuity.

2.3 Intermediate Value Theorem

Continuity plays an important role in the following theorem:

Theorem 13 (Intermediate Value Theorem)**.** *Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.*