Dr. Daniel Hast, *drhast@bu.edu*

Today's topics

1 Continuity

Briggs–Cochran–Gillett §2.6, pp. 103–115.

1.1 A couple more examples

Example 1. Determine whether f is continuous at $a = 3$.

$$f(x) = \begin{cases} \frac{x^2-4x+3}{x-3} & \text{if} \quad x \neq 3 \\ 2 & \text{if} \quad x = 3 \end{cases}$$

Example 2. Let

$$f(x) = \begin{cases} 2x & \text{if} \quad x < 1 \\ x^2 + 3x & \text{if} \quad x \geq 1 \end{cases}$$

1. Is f continuous at 1?
2. Is f continuous from the left or right at 1?
3. State the interval(s) of continuity.

1.2 Intermediate Value Theorem

Continuity plays an important role in the following theorem:

Theorem 3 (Intermediate Value Theorem)**.** *Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.*

2 Derivatives

Briggs–Cochran–Gillett §§3.1–3.2 pp. 131–152

In this section, we revisit the idea of finding the slope of a line tangent to a curve. We will:

- Identify the slope of the tangent line with the *instantaneous rate of change* of a function
- Study the slopes of the tangent lines as they change along a curve (these slopes are the values of a new function called the *derivative*)

2.1 Rate of change and the slope of the tangent line

Definition 4 (Rate of change and the slope of the tangent line)**.** *The average rate of change in f on the interval $[a, x]$ is the slope of the corresponding secant line:*

$$m_{sec} = \frac{f(x) - f(a)}{x - a}.$$

The instantaneous rate of change in f at a is

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

which is also the slope of the tangent line at $(a, f(a))$, provided this limit exists. The tangent line is the unique line through $(a, f(a))$ with slope m_{tan}. Its equation is $y - f(a) = m_{tan} \cdot (x - a)$.

Here is an alternative definition:

Definition 5 (Rate of change and the slope of the tangent line)**.** *The average rate of change in f on the interval $[a, a + h]$ is the slope of the corresponding secant line:*

$$m_{sec} = \frac{f(a + h) - f(a)}{h}.$$

The instantaneous rate of change in f at a is

$$m_{tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

which is also the slope of the tangent line at $(a, f(a))$, provided this limit exists.

Example 6. Let $f(x) = -3x^2 - 5x + 1$ and consider $P = (1, -7)$. Use Definition 4 to find the slope of the line tangent to the graph of f at P. Determine an equation of the tangent line at P.

The slope of the tangent line at $(1, -7)$ is given by the limit

$$\begin{aligned} m_{\tan} &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{-3x^2 - 5x + 1 - (-7)}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(-3x - 8)}{x - 1} \\ &= \lim_{x \to 1} (-3x - 8) = -11. \end{aligned}$$

So an equation of the tangent line is $y + 7 = -11(x - 1)$.

2.2 The derivative function

Definition 7. *The derivative of f is the function*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists and x is in the domain of f. If $f'(x)$ exists, we say that f is differentiable at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

Example 8. Consider the function $f(x) = 2x^3$.

(a) Compute the derivative f'.

(b) Find an equation of the tangent to the curve $y = f(x)$ at the point $(10, f(10))$.

Solutions:

(a) The derivative f' is given by

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)^3 - 2x^3}{h} \\ &= \lim_{h\to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 2x^3}{h} = \lim_{h\to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h} \\ &= \lim_{h\to 0}(6x^2 + 6xh + 2h^2) = 6x^2. \end{aligned}$$

(b) The slope of the tangent line is $f'(10) = 6 \cdot 10^2 = 600$, and $f(10) = 2 \cdot 10^3 = 2000$, so an equation of the tangent line is

$$y - 2000 = 600(x - 10).$$

Example 9. We compute the derivative of the function given by $f(x) = \sqrt{3x}$. First, observe that, for all $x > 0$ and all h such that $x + h \geq 0$, we have

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} = \frac{(\sqrt{3(x+h)} - \sqrt{3x})(\sqrt{3(x+h)} + \sqrt{3x})}{h(\sqrt{3(x+h)} + \sqrt{3x})} \\ &= \frac{3(x+h) - 3x}{h(\sqrt{3(x+h)} + \sqrt{3x})} = \frac{3h}{h(\sqrt{3(x+h)} + \sqrt{3x})} = \frac{3}{\sqrt{3(x+h)} + \sqrt{3x}}. \end{aligned}$$

(This algebraic technique is sometimes called "rationalizing the numerator".)

So for all $x > 0$, using the above computation, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)} + \sqrt{3x}} \\ &= \frac{3}{\lim_{h\to 0}(\sqrt{3(x+h)} + \sqrt{3x})} = \frac{3}{\sqrt{3x} + \sqrt{3x}} = \frac{3}{2\sqrt{3x}}. \end{aligned}$$