Dr. Daniel Hast, *drhast@bu.edu*

Today's topics

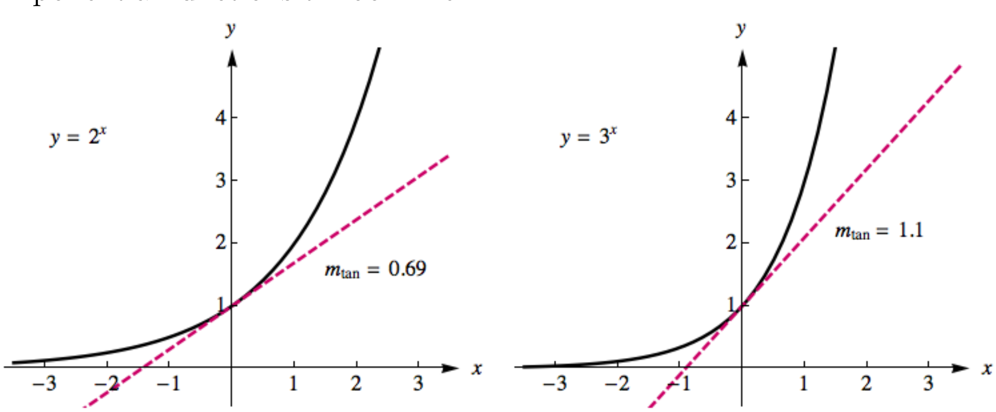
1 Rules of differentiation

Briggs–Cochran–Gillett §3.3, pp. 152–162

1.1 Derivative of e^x

Definition of e

Exponential functions b^x look like



The number e can be defined as the base needed in the exponential function to get the slope of the tangent to the graph at $x = 0$ equal to 1. We have $2.7182 < e < 2.7183$.

Definition 1. *e^x is the exponential function such that the slope of the tangent to the graph at $x = 0$ is 1, i.e.,*

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Derivative of e^x

Theorem 2. *The function $f(x) = e^x$ is differentiable for all real numbers x, and*

$$\frac{d}{dx}e^x = e^x.$$

Example 3. Find an equation of the tangent line to $y = \frac{e^x}{4} - x$ at $a = 0$. Then use a graphing utility to graph the curve and the tangent line on the same set of axes.

1.2 Higher order derivatives

Definition 4. *Assuming the function $f(x)$ can be differentiated as often as necessary, the* ***second derivative*** *of f is*

$$f''(x) = \frac{d}{dx}(f'(x)).$$

For integers $n \geq 2$, the **n*th derivative*** *of f is*

$$f^n(x) = \frac{d}{dx}(f^{(n-1)}(x)).$$

Example 5. Find $f'(x)$, $f''(x)$, and $f'''(x)$ for the following functions:

1. $f(x) = 3x^2 + 5e^x$
2. $f(x) = 10e^x$

2 The product and quotient rules

Briggs–Cochran–Gillett §3.4, pp. 163–170

We saw in the last class that the derivative of a sum of functions is the sum of the derivatives. So you might wonder about the derivative of a product of functions: is it the product of the derivatives? Consider $f(x) = x^3$ and $g(x) = x^4$; in this case, $\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(x^7) = 7x^6$ but $f'(x)g'(x) = 3x^2 \cdot 4x^3 = 12x^5$. Thus $\frac{d}{dx}(f(x)g(x)) \neq f'(x)g'(x)$ and likewise, the derivative of a quotient is not the quotient of the derivatives.

Today we discuss the rules for differentiating products and quotients of functions.

2.1 Product rule

Theorem 6 (Product rule)**.** *If f and g are differentiable at x, then*

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

2.2 Quotient rule

Theorem 7 (Quotient rule)**.** *If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of $\frac{f}{g}$ at x exists and*

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

2.3 Extended power rule

Theorem 8. *If r is any real number, then $\frac{d}{dx}(x^r) = rx^{r-1}$.*

2.4 Examples

Now we practice applying these rules.

Example 9. Compute the derivative of the following functions:

1. $g(x) = 6x - 2xe^x$

   Answer:

$$\begin{aligned} \frac{d}{dx}(6x - 2xe^x) &= 6 - \frac{d}{dx}(2xe^x) && \text{by sum, constant multiple, \& power rules} \\ &= 6 - \left(2x\frac{d}{dx}e^x + e^x\frac{d}{dx}(2x)\right) && \text{by product rule} \\ &= 6 - 2xe^x - 2e^x. \end{aligned}$$

2. $f(x) = (1 + \frac{1}{x^2})(x^2 + 1)$

   One approach:

$$\begin{aligned}\frac{d}{dx}\left[(1+x^{-2})(x^2+1)\right] &= (1+x^{-2})\frac{d}{dx}(x^2+1) + (x^2+1)\frac{d}{dx}(1+x^{-2}) \quad \text{by product rule} \\ &= (1+x^{-2})\cdot 2x + (x^2+1)(-2)x^{-3} \\ &= 2x - 2x^{-3}.\end{aligned}$$

   Alternatively, we can distribute first and then compute the derivative:

$$\frac{d}{dx}\left[(1+x^{-2})(x^2+1)\right] = \frac{d}{dx}\left(x^2+1+1+x^{-2}\right) = 2x - 2x^{-3}.$$

3. $f(x) = e^{-x}\sqrt{x}$

   Answer:

$$\begin{aligned}\frac{d}{dx}e^{-x}\sqrt{x} &= \frac{d}{dx}\frac{x^{1/2}}{e^x} \\ &= \frac{e^x \frac{d}{dx}x^{1/2} - x^{1/2}\frac{d}{dx}e^x}{(e^x)^2} \qquad \text{by quotient rule} \\ &= \frac{\frac{1}{2}e^x x^{-1/2} - x^{1/2}e^x}{e^{2x}} \\ &= \frac{1}{2e^x\sqrt{x}} - \frac{\sqrt{x}}{e^x}.\end{aligned}$$

Example 10. Let $y = \frac{x^2-2ax+a^2}{x-a}$, where a is a constant.

1. Use the quotient rule to find the derivative of the given function. Simplify your result.
2. Find the derivative by first simplifying the function. Verify that your answer agrees with part (1).

Example 11. Find the derivative of the following functions:

1. $f(w) = \dfrac{w^4 + 5w^2 + w}{w^2}$.
2. $f(x) = (1 - 2x)e^{-x}$

Example 12. Compute the derivative of $h(x) = \dfrac{x+1}{x^2 e^x}$.

Example 13. Suppose $f(2) = 2$ and $f'(2) = 3$. Let $g(x) = x^2 f(x)$ and $h(x) = \frac{f(x)}{x-3}$.

1. Find an equation of the line tangent to $y = g(x)$ at $x = 2$.
2. Find an equation of the line tangent to $y = h(x)$ at $x = 2$.