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Today's topics

Note: Solutions for quizzes 1 and 2 are available at https://math.bu.edu/people/tasso/ma123/

For sections B1-B5 (the Monday and Tuesday sections), all quiz 1 and 2 papers have been graded and were posted over the weekend on Gradescope. For the other sections (Wed-Thurs), most graded quiz 1 papers have already been posted as well, and the graded quiz 2 papers will be posted by late tomorrow afternoon. So, everyone will have their graded quiz 1 and 2 papers back before they take the double quiz (quiz 3) this week.

1 Derivatives of trigonometric functions

Briggs–Cochran–Gillett §3.5, pp. 171–178

1.1 Special trigonometric limits

In the same way that finding the derivative of e^x needed the special limit $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$, to find the derivatives of $\sin x$ and $\cos x$, we also need some limits that are not computed directly using the rules which we have already learned.

Theorem 1.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \textit{and} \quad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$$

Remark 2. In other words, $\frac{d\sin}{dx}(0) = 1$ and $\frac{d\cos}{dx}(0) = 0$. We can see this intuitively from the unit circle: at the point $(1, 0)$, corresponding to an angle of 0 radians, the line tangent to the unit circle is the vertical line $x = 1$. So, near this point, small changes in the angle produce an approximately equal change in the y-value (the sine value), but very little change in the x-coordinate (the cosine value).

Example 3 (§3.5, Ex. 17 and 20)**.** Use the theorem above to evaluate the following limits.

(a) $\lim_{x\to 0} \frac{\tan 7x}{\sin x}$

(b) $\lim_{x\to -3} \frac{\sin(x+3)}{x^2+8x+15}$

For the first limit:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(7x)}{\sin(x)} &= \lim_{x\to 0} \frac{\sin(7x)}{\sin(x)\cos(7x)} = \lim_{x\to 0} \frac{7x \cdot \sin(7x)}{7x \cdot \sin(x)\cos(7x)} \\
&= \lim_{x\to 0} \left(\frac{7}{\cos(7x)} \cdot \frac{\sin(7x)}{7x} \cdot \frac{x}{\sin(x)} \right) \\
&= \lim_{x\to 0} \frac{7}{\cos(7x)} \cdot \lim_{x\to 0} \frac{\sin(7x)}{7x} \cdot \left(\lim_{x\to 0} \frac{\sin(x)}{x} \right)^{-1} \\
&= \frac{7}{\cos(0)} \cdot 1 \cdot 1 = 7.
\end{aligned}$$

For the second limit:

$$\begin{aligned}
\lim_{x\to -3} \frac{\sin(x+3)}{x^2+8x+15} &= \lim_{x\to -3} \frac{\sin(x+3)}{(x+3)(x+5)} = \lim_{x\to -3} \frac{\sin(x+3)}{x+3} \cdot \lim_{x\to -3} \frac{1}{x+5} \\
&= 1 \cdot \frac{1}{-3+5} = \frac{1}{2}.
\end{aligned}$$

1.2 Derivatives

Theorem 4 (Derivatives of trigonometric functions)**.**

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\csc^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Example 5 (§3.5, Ex. 24 and 30)**.** Find dy/dx for the following functions.

(a) $y = 5x^2 + \cos x$

$$\frac{dy}{dx} = 10x - \sin(x)$$

(b) $y = \dfrac{1 - \sin x}{1 + \sin x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\frac{1-\sin x}{1+\sin x} = \frac{(1+\sin x)\frac{d}{dx}(1-\sin x) - (1-\sin x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} \\ &= \frac{(1+\sin x)(-\cos x) - (1-\sin x)(\cos x)}{(1+\sin x)^2} \\ &= \frac{-2\cos x}{(1+\sin x)^2}.\end{aligned}$$

Example 6 (§3.5, based on Ex. 46)**.** Find the derivative of $y = \dfrac{\tan w}{1 + \tan w}$

(a) ...directly using the formulas above;

(b) ...using only the formulas for the derivatives of $\sin x$ and $\cos x$.

Solutions:

(a) Using the formula for the derivative of tangent:

$$\begin{aligned}\frac{dy}{dw} &= \frac{(1+\tan w)\frac{d}{dw}(\tan w) - (\tan w)\frac{d}{dw}(1+\tan w)}{(1+\tan w)^2} \\ &= \frac{(1+\tan w)\sec^2 w - \tan w \sec^2 w}{(1+\tan w)^2} = \frac{\sec^2 w}{(1+\tan w)^2}.\end{aligned}$$

(b) Left as an exercise to the reader.

Example 7 (§3.5, Ex. 76)**.**

(a) For what values of x does $g(x) = x - \sin x$ have a horizontal tangent line?

(b) For what values of x does $g(x) = x - \sin x$ have a slope of 1?

This is equivalent to asking for solutions to the equations $g'(x) = 0$ and $g'(x) = 1$. Since

$$g'(x) = 1 - \cos(x),$$

for part (a) we want to solve $1 - \cos(x) = 0$, that is, $\cos(x) = 1$. This is the case exactly when x is an integer multiple of 2π. For part (b), we're solving $1 - \cos(x) = 1$, that is, $\cos(x) = 0$, which is the case exactly when $x = \frac{\pi}{2} + \pi k$ for an integer k.