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Today's topics

1 Derivatives as rates of change

Briggs–Cochran–Gillett §3.6, pp. 178–191

Definition 1 (Average and instantaneous velocity)**.** *Let $s = f(t)$ be the position function of an object moving along a line. The average velocity of the object over the time interval $[a, a + \Delta t]$ is the slope of the secant line between $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$:*

$$v_{av} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

The instantaneous velocity at a is the slope of the line tangent to the position curve, which is the derivative of the position function:

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

Definition 2 (Velocity, speed, and acceleration)**.** *Suppose an object moves along a line with position $s = f(t)$. Then*

- *the **velocity** at time t is $v = \frac{ds}{dt} = f'(t)$*
- *the **speed** at time t is $|v| = |f'(t)|$, and*
- *the **acceleration** at time t is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t)$.*

Example 3 (§3.6, based on Ex. 18)**.** Suppose the position of an object moving horizontally after t seconds is given by the function $s = f(t)$, where $f(t) = 6t - t^2$ with s measured in meters, with $s > 0$ corresponding to positions right of the origin. Consider the function on the interval $0 \leq t \leq 8$.

(a) Graph the position function.

(b) Find and graph the velocity function. When is the object stationary, moving to the right, and moving to the left?

We have $v(t) = 6 - 2t$. The object is stationary when $0 = v(t) = 6 - 2t$, so when $t = 3$. The object is moving to the right when $v(t) > 0$, i.e., when $0 \leq t < 3$, and is moving to the left when $3 < t \leq 8$.

(c) Determine the velocity and acceleration of the object at $t = 1$.

We have $v(1) = 6 - 2 = 4$ and $a(1) = -2$.

(d) Determine the acceleration of the object when its velocity is zero.

The acceleration is always -2 because $dv/dt = -2$.

(e) On what intervals is the speed increasing?

The speed is $|v(t)| = |6 - 2t|$, which is increasing when $t > 3$.

Example 4 (§3.6, Ex. 24)**.** Suppose a stone is thrown vertically upward from the edge of a cliff on Earth (where the acceleration due to gravity is about $9.8\,\text{m/s}^2$) with an initial velocity of 19.6 m/s from height of 24.5 m above the ground. The height (in meters) s of the stone above the ground after t seconds is given by $s(t) = -4.9t^2 + 19.6t + 24.5$.

(a) Determine the velocity v of the stone after t seconds.

$$v(t) = -9.8t + 19.6$$

(b) When does the stone reach its highest point?

Set $v(t) = 0$ and solve for t.

(c) What is the height of the stone at the highest point?

Plug the t value from (b) into s to get the height.

(d) When does the stone strike the ground?

Factoring, we have $s(t) = -4.9(t + 1)(t - 5)$, so this is when $t = -1$ or $t = 5$. Only $t = 5$ makes physical sense ($t = -1$ is before the stone was thrown), so that's the answer.

(e) With what velocity does the stone strike the ground?

Evaluate $v(5)$.

(f) On what intervals is the speed increasing?

The speed is $|-9.8t + 19.6|$, which is increasing when $t > 2$.

2 Chain Rule

Briggs–Cochran–Gillett §3.7, pp. 191–200

The *Chain Rule* lets us differentiate composite functions:

Theorem 5 (Chain Rule)**.** *Suppose $y = f(u)$ is differentiable at $u = g(x)$, and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways:*

- *Version 1:*
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
- *Version 2:*
$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Example 6 (§3.7, Ex. 16)**.** Use the Chain Rule to compute the derivative of

$$y = (5x^2 + 11x)^{4/3}.$$

First, let's write down the component functions and compute their derivatives:

$$y = f(u) = u^{4/3} \qquad \frac{dy}{du} = f'(u) = \frac{4}{3}u^{1/3}$$
$$u = g(x) = 5x^2 + 11x \qquad \frac{du}{dx} = g'(x) = 10x + 11$$

Now we apply the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{4}{3}u^{1/3} \cdot (10x + 11) = \frac{4}{3}(5x^2 + 11x)^{1/3} \cdot (10x + 11).$$

Equivalently, using the notation of version 2 of the chain rule:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = \frac{4}{3}g(x)^{1/3} \cdot (10x + 11) = \frac{4}{3}(5x^2 + 11x)^{1/3} \cdot (10x + 11).$$