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Today's topics

1 Chain Rule

Briggs–Cochran–Gillett §3.7, pp. 191–200

Recall from last time:

Theorem 1 (Chain rule)**.** *If f and g are differentiable functions, then the composite function $f \circ g$ is differentiable and*

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Example 2 (§3.7, based on Ex. 18)**.** Use the Chain Rule to compute the derivatives of the following functions:

(a) $y = \cos(x^5)$

(b) $y = \cos^5(x)$

We have

$$\frac{d}{dx}\cos(x^5) = -\sin(x^5) \cdot \frac{d}{dx}x^5 = -\sin(x^5) \cdot 5x^4,$$
$$\frac{d}{dx}\cos^5(x) = \frac{d}{dx}(\cos x)^5 = 5(\cos x)^4 \cdot \frac{d}{dx}\cos x = 5\cos^4(x) \cdot (-\sin x).$$

Example 3 (§3.7, Ex. 26)**.** Let $h(x) = f(g(x))$ and $k(x) = g(g(x))$. Use the table to compute the following derivatives.

(a) $h'(1)$

(b) $h'(2)$

(c) $h'(3)$

(d) $k'(3)$

(e) $k'(1)$

(f) $k'(5)$

x	1	2	3	4	5
$f'(x)$	−6	−3	8	7	2
$g(x)$	4	1	5	2	3
$g'(x)$	9	7	3	−1	−5

Let's just compute a couple of these here:

$$h'(1) = f'(g(1))g'(1) = f'(4)g'(1) = 7 \cdot 9 = 63,$$
$$k'(3) = g'(g(3))g'(3) = g'(5)g'(3) = -5 \cdot 3 = -15.$$

Example 4 (§3.7, Ex. 19, 36)**.** Calculate the derivatives of the following functions:

(a) $y = \sqrt{x^2+1}$

(b) $y = e^{\tan t}$

Answers:

(a) $\frac{dy}{dx} = \frac{1}{2}(x^2+1)^{-1/2} \cdot (2x)$.

(b) $\frac{dy}{dx} = e^{\tan t} \cdot \sec^2 t$.

Example 5 (§3.7, Ex. 54, 58, 60, 70, 74)**.** Calculate the derivatives of the following functions:

(a) $y = \sin^2(e^{3x+1})$

$$\frac{dy}{dx} = 2\sin(e^{3x+1}) \cdot \cos(e^{3x+1}) \cdot e^{3x+1} \cdot 3.$$

(b) $y = (1 - e^{-0.05x})^{-1}$

$$\frac{dy}{dx} = -(1 - e^{-0.05x})^{-2} \cdot -e^{-0.05x} \cdot (-0.05).$$

(c) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

(d) $y = e^{2x}(2x-7)^5$

(e) $y = (2z+5)^{1.75} \tan z$

For (c), it's helpful to first compute the derivative of the inner function:

$$\frac{d}{dx}\left(x + \sqrt{x + \sqrt{x}}\right) = 1 + \frac{1}{2}(x + \sqrt{x})^{-1/2} \cdot \left(1 + \frac{1}{2}x^{-1/2}\right),$$

so

$$\begin{aligned} \frac{d}{dx}\sqrt{x + \sqrt{x + \sqrt{x}}} &= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2} \cdot \frac{d}{dx}\left(x + \sqrt{x + \sqrt{x}}\right) \\ &= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2} \cdot \left[1 + \frac{1}{2}(x + \sqrt{x})^{-1/2} \cdot \left(1 + \frac{1}{2}x^{-1/2}\right)\right]. \end{aligned}$$

For (d), we apply the product rule first:

$$\frac{d}{dx}e^{2x}(2x-7)^5 = 2e^{2x}\cdot(2x-7)^5 + e^{2x}\cdot 5(2x-7)^4\cdot 2.$$

Part (e) is similar.

Example 6 (§3.7, Ex. 96)**.** Suppose f is differentiable on $[-2,2]$ with $f'(0)=3$ and $f'(1)=5$. Let $g(x)=f(\sin x)$. Evaluate the following expressions:

(a) $g'(0)$

(b) $g'(\frac{\pi}{2})$

(c) $g'(\pi)$