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Today's topics

1 Chain Rule

Example 1 (§3.7, Ex. 96)**.** Suppose f is differentiable on $[-2, 2]$ with $f'(0) = 3$ and $f'(1) = 5$. Let $g(x) = f(\sin x)$. Evaluate the following expressions:

(a) $g'(0)$

(b) $g'(\frac{\pi}{2})$

(c) $g'(\pi)$

Using the chain rule, $g'(x) = f'(\sin x)\cos x$. So

$$\begin{aligned} g'(0) &= f'(\sin 0)\cos 0 = f'(0)\cdot 1 = 3, \\ g'(\tfrac{\pi}{2}) &= f'(\sin\tfrac{\pi}{2})\cos\tfrac{\pi}{2} = f'(1)\cdot 0 = 0, \\ g'(\pi) &= f'(\sin\pi)\cos\pi = f'(0)\cdot(-1) = -3. \end{aligned}$$

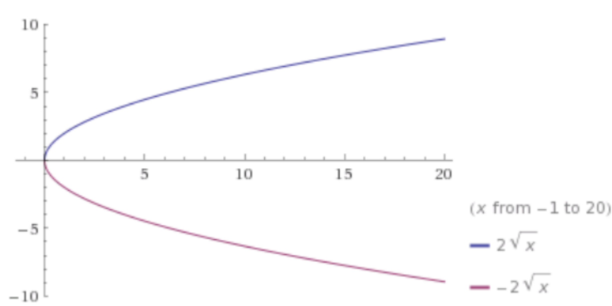
2 Implicit differentiation

Briggs–Cochran–Gillett §3.8, pp. 201–208

2.1 What it is and how to do it

Consider the curve defined by $y^2 = 4x$, for $x \in \mathbb{R}$. To represent it in the xy-plane we need to solve for x:

$$\begin{aligned} y^2 &= 4x \\ \iff y &= \pm\sqrt{4x} \end{aligned}$$



So y is not a function of x, because for each value of x there are 2 values of y. But the original expression does define a curve in the plane which is the union of the graphs of the two functions $y = \sqrt{4x}$ and $y = -\sqrt{4x}$. We say in this case that y is **implicitly defined** by the expression $y^2 = 4x$.

If we want to find slopes of lines tangent to this curve we need to differentiate the original expression. In this example we could solve it for y and then differentiate. But in many examples this is not possible. So we want to **implicitly** differentiate the original expression to get an expression for dy/dx. We have

$$(y(x))^2 = 4x.$$

Hence

$$\frac{d}{dx}(y(x))^2 = \frac{d}{dx}4x = 4.$$

By the chain rule, we also have

$$\frac{d}{dx}(y(x))^2 = 2y(x)\frac{d}{dx}y(x),$$

so if $y \neq 0$, we obtain

$$\frac{d}{dx}y(x) = \frac{4}{2y} = \frac{2}{y}.$$

Note that the derivative $\frac{d}{dx}(y(x))$ is not defined at $y = 0$: that corresponds to the point where the curve has a vertical tangent line, hence its slope is not defined. At any other point we can find the slope of the tangent line. For example, the slope of the tangent line at $(1, 2)$ (which is a point on the curve) will be

$$y'(2) = \frac{2}{y(1)} = \frac{2}{2} = 1.$$

2.2 Examples

Example 2 (§3.8, Ex. 20)**.** Consider the curve given by $\tan(xy) = x + y$. Use implicit differentiation to find dy/dx and find the slope of the curve at the point $(0, 0)$.

Differentiating both sides, we obtain

$$\frac{d}{dx}\tan(xy) = \frac{d}{dx}(x + y).$$

We compute each side:

$$\frac{d}{dx}\tan(xy) = \sec^2(xy)\frac{d}{dx}(xy) = \sec^2(xy)\left(y + x\frac{dy}{dx}\right),$$
$$\frac{d}{dx}(x + y) = 1 + \frac{dy}{dx}.$$

Thus

$$\sec^2(xy)\left(y + x\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}.$$

To find the slope at $(0, 0)$, we substitute $x = y = 0$ into the above:

$$\sec^2(0)\left(0 + 0\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}.$$

So $dy/dx = -1$.

Example 3 (Learning Catalytics question; §3.8, Ex. 26)**.** Determine the slope of the curve given by the equation $(x + y)^{2/3} = y$ at the point $(4, 4)$.

Differentiating both sides with respect to x, we obtain

$$\frac{2}{3}(x + y)^{-1/3}\left(1 + \frac{dy}{dx}\right) = \frac{dy}{dx}.$$

Substituting $x = y = 4$, we obtain

$$\frac{2}{3}(4 + 4)^{-1/3}\left(1 + \frac{dy}{dx}\right) = \frac{dy}{dx}.$$

Note that $8^{-1/3} = 1/2$. Solving the above linear equation yields $dy/dx = 1/2$.

Example 4 (§3.8, Ex. 32, 36, and 40)**.** Use implicit differentiation to find dy/dx for the curves given by the following equations:

(a) $e^{xy} = 2y$

(b) $(xy + 1)^3 = x - y^2 + 8$

(c) $\sqrt{x + y^2} = \sin y$

In each case we differentiate both sides, then solve for dy/dx:

(a)

$$\begin{aligned} e^{xy}(y + x\frac{dy}{dx}) &= 2\frac{dy}{dx} \\ (xe^{xy} - 2)\frac{dy}{dx} &= -ye^{xy} \\ \frac{dy}{dx} &= \frac{-ye^{xy}}{xe^{xy} - 2}. \end{aligned}$$

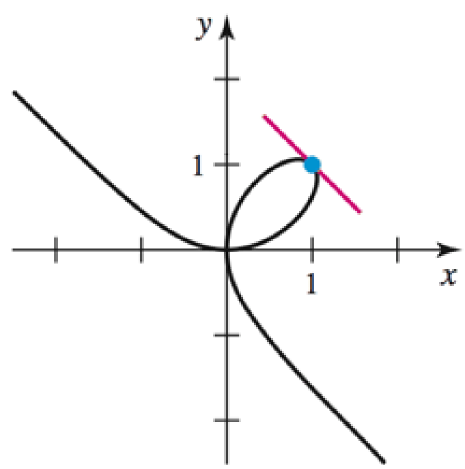
(b)

$$\begin{aligned} 3(xy+1)^2\left(y + x\frac{dy}{dx}\right) &= 1 - 2y\frac{dy}{dx} \\ 3(xy+1)^2 x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 1 - 3(xy+1)^2 y \\ \frac{dy}{dx} &= \frac{1-3y(xy+1)^2}{3x(xy+1)^2+2y}. \end{aligned}$$

(c)

$$\begin{aligned} \frac{1}{2}(x+y^2)^{-1/2}(1+2y\frac{dy}{dx}) &= \cos(y)\frac{dy}{dx} \\ \frac{1}{2}(x+y^2)^{-1/2}(1+2y\frac{dy}{dx}) &= \cos(y)\frac{dy}{dx} \\ y(x+y^2)^{-1/2}\frac{dy}{dx} - \cos(y)\frac{dy}{dx} &= -\frac{1}{2}(x+y^2)^{-1/2} \\ \frac{dy}{dx} &= \frac{-\frac{1}{2}(x+y^2)^{-1/2}}{y(x+y^2)^{-1/2} - \cos(y)}. \end{aligned}$$

Example 5 (§3.8, Ex. 46)**.** Consider the curve defined by $x^3 + y^3 = 2xy$.



Verify that the point $(1,1)$ lies on the curve and determine an equation of the line tangent to the curve at that point.

We have $1^3 + 1^3 = 2 \cdot 1 \cdot 1$, so $(1,1)$ lies on the curve. Using implicit differentiation, we have

$$3x^2 + 3y^2\frac{dy}{dx} = 2y + 2x\frac{dy}{dx},$$

so if $x = y = 1$, then

$$3 + 3\frac{dy}{dx} = 2 + 2\frac{dy}{dx}.$$

Solving this linear equation yields $\frac{dy}{dx} = -1$. So the tangent line to the curve at $(1, 1)$ is given by the equation

$$y - 1 = -1 \cdot (x - 1).$$

Example 6 (§3.8, Ex. 42)**.** The lateral surface area of a cone of radius r and height h (the surface area excluding the base) is $A = \pi r\sqrt{r^2 + h^2}$.

(a) Find dr/dh for a cone with a lateral surface area of $A = 1500\pi$.

(b) Evaluate this derivative when $r = 30$ and $h = 40$.

We'll do this problem in class next time.