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Today's topics

1 Implicit differentiation

Briggs–Cochran–Gillett §3.8, pp. 201–208

Example 1 (§3.8, Ex. 42)**.** The lateral surface area of a cone of radius r and height h (the surface area excluding the base) is $A = \pi r\sqrt{r^2 + h^2}$.

(a) Find dr/dh for a cone with a lateral surface area of $A = 1500\pi$.

(b) Evaluate this derivative when $r = 30$ and $h = 40$.

Solution: We use implicit differentiation on the surface area formula. Note that $A = 1500\pi$ is constant, so using rules for differentiation, we obtain

$$0 = \frac{d}{dh}\pi r\sqrt{r^2 + h^2} = \left(\pi\frac{dr}{dh}\sqrt{r^2 + h^2} + \pi r \cdot \frac{1}{2}(r^2 + h^2)^{-1/2} \cdot (2r\frac{dr}{dh} + 2h)\right).$$

Dividing both sides by π and multiplying both sides by $\sqrt{r^2 + h^2}$ yields

$$0 = (r^2 + h^2)\frac{dr}{dh} + \frac{r}{2}\left(2r\frac{dr}{dh} + 2h\right) = (2r^2 + h^2)\frac{dr}{dh} + rh.$$

Thus,

$$\frac{dr}{dh} = \frac{-rh}{2r^2 + h^2}.$$

Substituting $r = 30$ and $h = 40$, we get

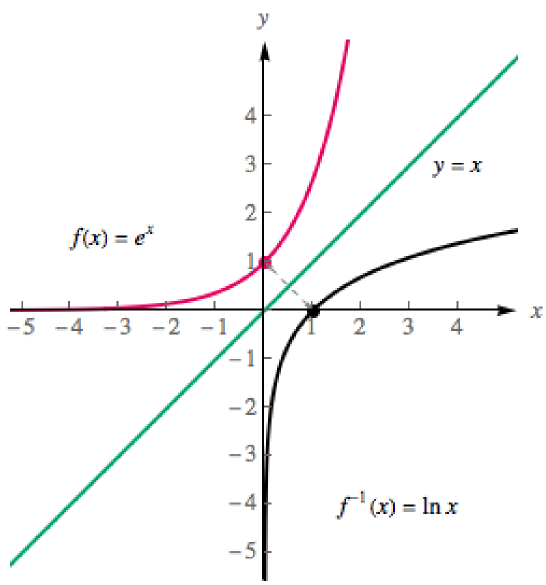
$$\frac{dr}{dh} = \frac{-30 \cdot 40}{2 \cdot 30^2 + 40^2} = \frac{-1200}{3400} = \frac{-6}{17}.$$

2 Derivatives of logarithmic and exponential functions

Briggs–Cochran–Gillett §3.9, pp. 208–218

2.1 Derivatives

Recall that the exponential function $f(x) = e^x$ is a one-to-one function on the interval $(-\infty, \infty)$. Thus it has an inverse, which is the natural logarithmic function $f^{-1}(x) = \ln(x)$. The graphs of f and f^{-1} are symmetric about the line $y = x$, as below.



We summarize the inverse properties for e^x and $\ln x$ below:

(1) $e^{\ln x} = x$ for $x > 0$ and $\ln(e^x) = x$ for all x.

(2) $y = \ln x$ if and only if $x = e^y$.

(3) For real numbers x and $b > 0$, we have $b^x = e^{\ln b^x} = e^{x \ln b}$.

We can use implicit differentiation to calculate the derivative of $\ln x$: If $\ln x = y$, then $x = e^y$, so

$$1 = \frac{d}{dx}x = \frac{d}{dx}e^y = e^y \cdot \frac{dy}{dx} = x \cdot \frac{dy}{dx}.$$

This yields the following:

Theorem 2 (Derivative of $\ln x$)**.** *We have*

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \text{ for } x > 0 \text{ and } \frac{d}{dx}(\ln|x|) = \frac{1}{x} \text{ for } x \neq 0.$$

If u is differentiable at x and $u(x) \neq 0$, then

$$\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}.$$

The property $e^{x \ln b} = b^x$ results in the following theorem:

Theorem 3 (Derivative of b^x). *If $b > 0$ and $b \neq 1$, then for all x, we have*

$$\frac{d}{dx}(b^x) = b^x \ln b.$$

We can also use this to extend the power rule to all real exponents:

Theorem 4 (General power rule). *For real numbers p and for $x > 0$, we have*

$$\frac{d}{dx}(x^p) = px^{p-1}.$$

Furthermore, if u is a positive differentiable function on its domain, then

$$\frac{d}{dx}(u(x)^p) = p(u(x))^{p-1} \cdot u'(x).$$

We also use implicit differentiation to calculate the derivative of $\log_b x$:

Theorem 5 (Derivative of $\log_b x$). *If $b > 0$ and $b \neq 1$, then*

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}, \text{ for } x > 0 \text{ and } \frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b}, \text{ for } x \neq 0.$$

2.2 Exercises

Example 6. Find the following derivatives.

1. $\frac{d}{dx} \ln(7x)$
2. $\frac{d}{dx} \ln(2x^8)$
3. $\frac{d}{dx} \ln|x^2 - 1|$
4. $\frac{d}{dx} \ln(\cos^2 x)$

Solutions:

1. Using the chain rule directly:

$$\frac{d}{dx} \ln(7x) = \frac{1}{7x} \cdot 7 = \frac{1}{x}.$$

Or, using properties of logarithms:

$$\frac{d}{dx} \ln(7x) = \frac{d}{dx} (\ln(7) + \ln(x)) = 0 + \frac{1}{x} = \frac{1}{x}.$$

2. Using the chain rule:
$$\frac{d}{dx}\ln(2x^8) = \frac{1}{2x^8}\cdot 16x^7 = \frac{8}{x}.$$
Using properties of logarithms:
$$\frac{d}{dx}\ln(2x^8) = \frac{d}{dx}\left(\ln(2) + 8\ln|x|\right) = 0 + 8\cdot\frac{1}{x} = \frac{8}{x}.$$
Note: the absolute value is there to ensure that the domain remains the same (since $2x^8$ is positive for all $x \neq 0$).

3. $$\frac{d}{dx}\ln|x^2-1| = \frac{2x}{x^2-1}.$$

4. Using the chain rule directly:
$$\frac{d}{dx}\ln(\cos^2 x) = \frac{1}{\cos^2 x}\cdot 2\cos x\cdot(-\sin x) = \frac{-2\sin x}{\cos x} = -2\tan x.$$
Alternately:
$$\frac{d}{dx}\ln(\cos^2 x) = \frac{d}{dx}2\ln|\cos x| = \frac{2}{\cos x}\cdot(-\sin x) = -2\tan x.$$

Next time, we'll see more examples involving derivatives of logarithmic and exponential functions, as well as derivatives of inverse trigonometric functions.