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Today's topics

1 Derivatives of logarithmic and exponential functions

Briggs–Cochran–Gillett §3.9, pp. 208–218

Example 1. Compute the derivative $\frac{d}{dx}(x^\pi + \pi^x)$.

We have

$$\frac{d}{dx}(x^\pi + \pi^x) = \pi x^{\pi-1} + \pi^x \ln \pi.$$

Example 2 (§3.9, Ex. 90)**.** Compute the following higher order derivatives: $\frac{d^n}{dx^n}(2^x)$.

We have $\frac{d}{dx}2^x = 2^x \ln 2$, so

$$\frac{d^n}{dx^n}2^x = 2^x(\ln 2)^n.$$

Example 3 (§3.9, Ex. 70)**.** Let $f(x) = \ln \frac{2x}{(x^2+1)^3}$. Use the properties of logarithms to simplify the function before computing $f'(x)$.

Solution: Using properties of logarithms, we have

$$f(x) = \ln(2x) - \ln((x^2+1)^3) = \ln(2) + \ln(x) - 3\ln(x^2+1).$$

At this point, we cannot simplify any further using properties of logarithms. Taking the derivative, we obtain

$$f'(x) = \frac{1}{x} - 3 \cdot \frac{1}{x^2+1} \cdot 2x = \frac{1}{x} - \frac{6x}{x^2+1}.$$

Example 4 (§3.9, Ex. 60)**.** Determine whether the graph of $y = x^{\sqrt{x}}$ has any horizontal tangent lines.

Solution: Using properties of logarithms and exponentials, we have

$$x^{\sqrt{x}} = e^{\ln(x^{\sqrt{x}})} = e^{\sqrt{x}\ln x}.$$

Thus,

$$\begin{aligned}\frac{dy}{dx} &= e^{\sqrt{x}\ln x} \cdot \frac{d}{dx}\left(\sqrt{x}\ln x\right) = e^{\sqrt{x}\ln x} \cdot \left(\frac{1}{2\sqrt{x}} \cdot \ln x + \sqrt{x} \cdot \frac{1}{x}\right) \\ &= e^{\sqrt{x}\ln x} \cdot \frac{1}{\sqrt{x}} \cdot \left(\frac{\ln x}{2} + 1\right).\end{aligned}$$

So we are looking for solutions to the equation

$$0 = e^{\sqrt{x}\ln x} \cdot \frac{1}{\sqrt{x}} \cdot \left(\frac{\ln x}{2} + 1\right).$$

Note that $e^{\sqrt{x}\ln x}$ is always positive for $x > 0$, and likewise for $1/\sqrt{x}$. So we can divide both sides by these terms, yields

$$0 = \frac{\ln x}{2} + 1,$$

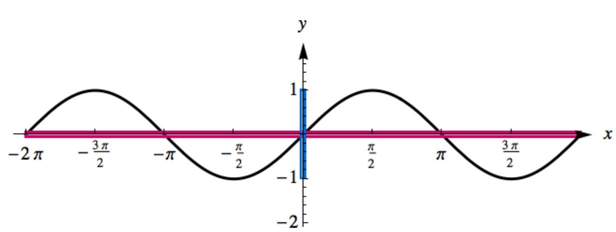
which has the unique solution $\ln x = -2$, or equivalently, $x = e^{-2}$. This is the unique x-value at which the graph of $y = x^{\sqrt{x}}$ has a horizontal tangent line.

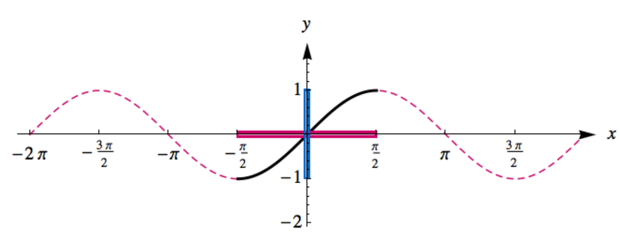
2 Review of Inverse Trigonometric Functions

Briggs–Cochran–Gillett §1.4, pp. 39–51

2.1 Sine and Arcsine

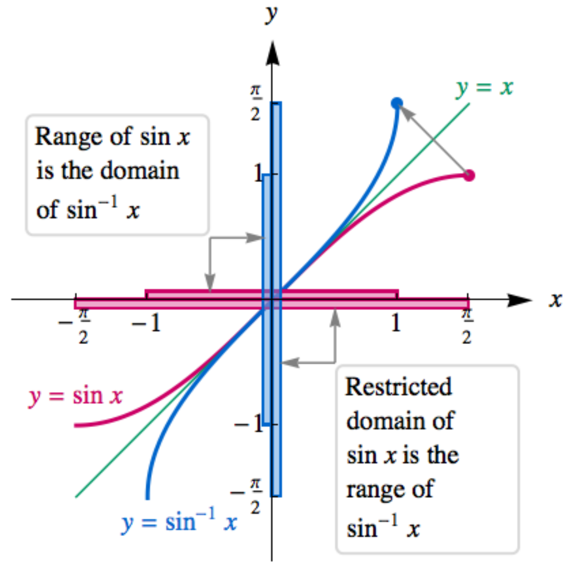
To invert a function f on a domain we need it to be one-to-one on that domain. This means that every output of the function f must correspond to exactly one input. (Recall that the one-to-one property is checked graphically by using the *horizontal line test*.) The function $\sin x$ is not one-to-one over all its domain, but if we restrict it to $[-\pi/2, \pi/2]$ it is one-to-one, and it makes sense to talk about its inverse.





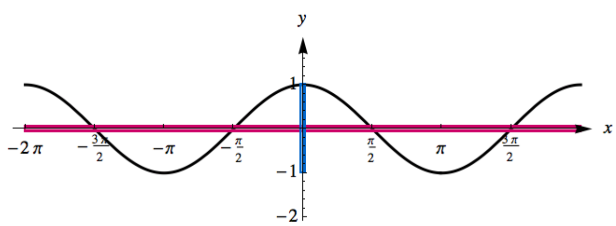
The inverse of $\sin x$ is $\arcsin x = \sin^{-1} x$.

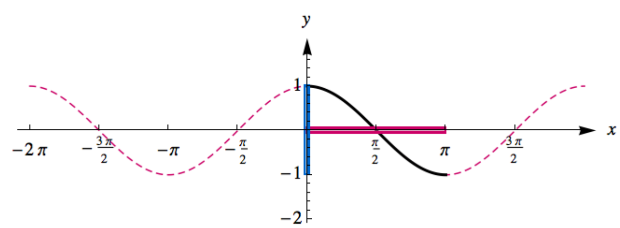
- $\sin^{-1}(x)$ is the angle whose sin is x
- Domain($\sin^{-1} x$)=$[-1, 1]$ (range of $\sin x$)
- Range($\sin^{-1} x$)=$[-\pi/2, \pi/2]$ (restricted domain of $\sin x$)
- Graphically the two functions are symmetric about the line $y = x$
- $\sin(\sin^{-1}(x)) = x$ for all x in $[-1, 1]$
- $\sin^{-1}(\sin(x)) = x$ for all x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- Remark: $\sin^{-1} x$ **is not** $\dfrac{1}{\sin x}$



2.2 Cosine and Arccosine

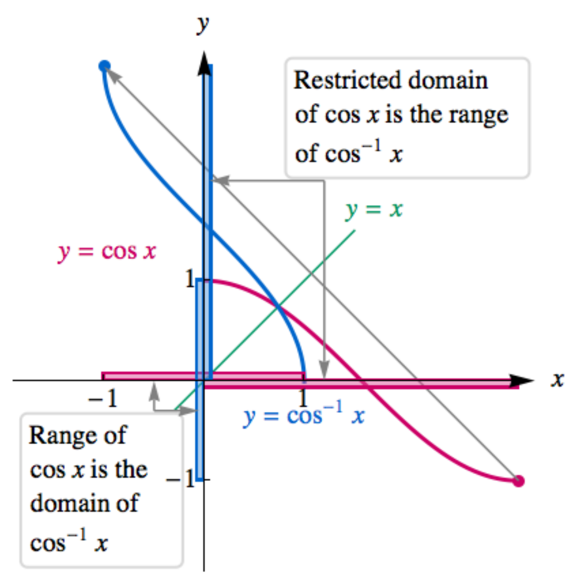
In the same way as above, the function $\cos x$ is not one-to-one over all its domain, but if we restrict it to $[0, \pi]$ it is one-to-one, and it makes sense to talk about its inverse.





The inverse of $\cos x$ is $\arccos x = \cos^{-1} x$.

- $\cos^{-1}(x)$ is the angle whose cos is x
- Domain ($\cos^{-1} x$)=$[-1, 1]$ (range of $\cos x$)
- Range ($\cos^{-1} x$)=$[0, \pi]$ (restricted domain of $\cos x$)
- Graphically the two functions are symmetric about the line $y = x$
- $\cos(\cos^{-1}(x)) = x$ for all x in $[-1, 1]$
- $\cos^{-1}(\cos(x)) = x$ for all x in $[0, \pi]$
- Remark: $\cos^{-1} x$ **is not** $\dfrac{1}{\cos x}$



2.3 Other inverse trig functions

We proceed in the same way to find the inverse functions to all trigonometric functions.

DEFINITION **Other Inverse Trigonometric Functions**

$y = \tan^{-1} x$ is the value of y such that $x = \tan y$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$y = \cot^{-1} x$ is the value of y such that $x = \cot y$, where $0 < y < \pi$.

The domain of both $\tan^{-1} x$ and $\cot^{-1} x$ is $\{x: -\infty < x < \infty\}$.

$y = \sec^{-1} x$ is the value of y such that $x = \sec y$, where $0 \leq y \leq \pi$, with $y \neq \frac{\pi}{2}$.

$y = \csc^{-1} x$ is the value of y such that $x = \csc y$, where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, with $y \neq 0$.

The domain of both $\sec^{-1} x$ and $\csc^{-1} x$ is $\{x: |x| \geq 1\}$.

2.4 Examples

Example 5 (§1.4, Ex. 49, 54, 55, 75, 76, 77)**.** Evaluate the following expressions (without a calculator!) or state that they are not defined.

(a) $\sin^{-1}(1)$

(b) $\cos^{-1}(2)$

(c) $\cos^{-1}(-1/2)$

(d) $\tan^{-1}(\sqrt{3})$

(e) $\cot^{-1}(-1/\sqrt{3})$

(f) $\sec^{-1}(2)$

Example 6 (§1.4, Ex. 61, 63, 83)**.** Simplify the given expressions. Assume $x > 0$. (Hint: draw a relevant right triangle in the unit circle.)

(a) $\cos(\sin^{-1} x)$

(b) $\sin(\cos^{-1}(x/2))$

(c) $\cos(\tan^{-1} x)$