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Today's topics

1 Derivatives of Inverse Trigonometric Functions 1

1 Derivatives of Inverse Trigonometric Functions

Example 1 (§1.4, Ex. 61, 63, 83)**.** Simplify the given expressions. Assume $x > 0$. (Hint: draw a relevant right triangle.)

(a) $\cos(\sin^{-1} x)$ (b) $\sin(\cos^{-1}(x/2))$ (c) $\cos(\tan^{-1} x)$

Solutions:

(a) Drawing a right triangle with hypotenuse 1, height x, and base $\cos(\sin^{-1} x)$, we see that

$$\cos(\sin^{-1} x) = \sqrt{1 - x^2}.$$

(b) Draw a right triangle with hypotenuse 1, base $x/2$, and height $\sin(\cos^{-1}(x/2))$:

$$\sin(\cos^{-1}(x/2)) = \sqrt{1 - (x/2)^2}.$$

(c) Draw a right triangle with hypotenuse 1, base $\cos(\tan^{-1} x)$ and height $x \cos(\tan^{-1} x)$ (to ensure that the ratio of the side lengths is x):

$$\left(x \cos(\tan^{-1} x)\right)^2 + \left(\cos(\tan^{-1} x)\right)^2 = 1,$$

so

$$(x^2 + 1) \left(\cos(\tan^{-1} x)\right)^2 = 1.$$

Thus

$$\cos(\tan^{-1} x) = \frac{1}{\sqrt{x^2 + 1}}.$$

Briggs–Cochran–Gillett §3.10, pp. 218–227

Let's use implicit differentiation to compute the derivative of $\sin^{-1} x$: If $y = \sin^{-1} x$, then $\sin(y) = x$, so

$$\frac{d}{dx} \sin y = \frac{d}{dx} x = 1.$$

We compute $\frac{d}{dx} \sin y$ using the chain rule:

$$\frac{d}{dx} \sin(y) = \cos(y) \cdot \frac{dy}{dx}.$$

By the Pythagorean theorem, if $\cos y \geq 0$, then

$$\cos y = \sqrt{1 - \sin^2 y}.$$

Since $y = \sin^{-1} x$ is in the interval $[-\pi/2, \pi/2]$, we do indeed have $\cos y \geq 0$. So

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

Similarly, for the other inverse trigonometric functions, using implicit differentiation and trigonometric identities we get:

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2}, \text{ for } -\infty < x < \infty$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2 - 1}}, \text{ for } |x| > 1$$

Example 2 (§3.10, Ex. 14, 22, 24, 30, 39)**.** Evaluate the derivatives of the following functions.

1. $f(x) = x \sin^{-1} x$

$$\frac{df}{dx} = \sin^{-1} x + x \cdot \frac{1}{\sqrt{1 - x^2}}.$$

2. $g(z) = \tan^{-1}(1/z)$

$$\frac{dg}{dz} = \frac{1}{1 + (1/z)^2} \cdot \frac{-1}{z^2} = \frac{-1}{z^2 + 1}.$$

   Note that g has the same derivative as the inverse cotangent. This is because $\cot^{-1}(x) = \tan^{-1}(1/x)$ for all x, so in fact g *is* the inverse cotangent function.

3. $f(x) = \sec^{-1}\sqrt{x}$: For all $x > 1$ (the domain of f), we have:

$$f'(x) = \frac{1}{|\sqrt{x}|\sqrt{\sqrt{x}^2 - 1}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x\sqrt{x-1}}.$$

4. $f(t) = (\cos^{-1} t)^2$

$$f'(t) = 2(\cos^{-1} t) \cdot \frac{-1}{\sqrt{1-x^2}} = \frac{-2\cot^{-1} t}{\sqrt{1-x^2}}.$$

5. $f(s) = \cot^{-1}(e^s)$

$$\frac{df}{ds} = \frac{-1}{1+(e^s)^2} \cdot e^s = \frac{-e^s}{1+e^{2s}}.$$

Example 3 (§3.10, Ex. 34, 38)**.** Evaluate the derivatives of the following functions.

1. $f(w) = \sin(\sec^{-1}(2w))$

$$\frac{df}{dw} = \cos(\sec^{-1}(2w)) \cdot \frac{1}{|2w|\sqrt{(2w)^2-1}} \cdot 2 = \frac{1}{2w} \cdot \frac{2}{|2w|\sqrt{(2w)^2-1}},$$

   using the fact that $\cos(\sec^{-1} x) = \cos(\cos^{-1}(1/x)) = 1/x$ for all x with $|x| > 1$.

2. $f(x) = \sin(\tan^{-1}(\ln x))$

$$\frac{df}{dx} = \cos(\tan^{-1}(\ln x)) \cdot \frac{1}{1+(\ln x)^2} \cdot \frac{1}{x} = \frac{1}{\sqrt{(\ln x)^2+1}} \cdot \frac{1}{1+(\ln x)^2} \cdot \frac{1}{x}$$
$$= \frac{1}{x\left((\ln x)^2+1\right)^{3/2}}.$$

Example 4 (§3.10, Ex. 44)**.** Find an equation of the line tangent to the graph of $f(x) = \sec^{-1}(e^x)$ at the point $(\ln 2, \pi/3)$.

The derivative is

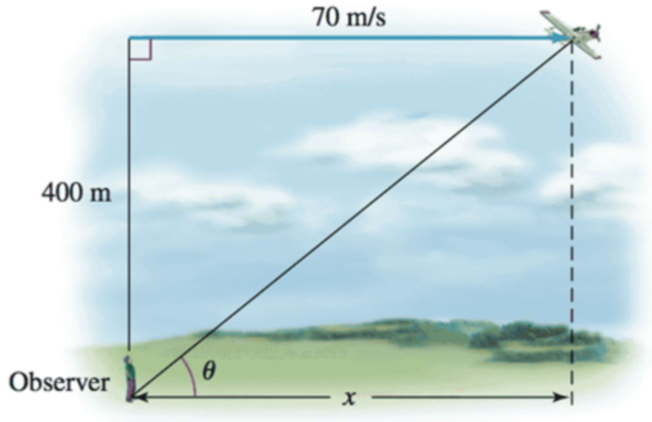
$$f'(x) = \frac{1}{|e^x|\sqrt{(e^x)^2-1}} \cdot e^x = \frac{1}{\sqrt{e^{2x}-1}}.$$

Since $e^{2\ln 2} = e^{\ln(2^2)} = 4$, we have $f'(\ln 2) = 1/\sqrt{3}$, so an equation of the tangent line is

$$y - \frac{\pi}{3} = \frac{1}{\sqrt{3}}(x - \ln 2).$$

1.1 Application

Example 5 (§3.10, Ex. 46)**.** A small plane, moving at 70 m/s, flies horizontally on a line 400 m directly above an observer. Let θ be the angle of elevation of the plane (see figure).



(a) What is the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when the plane is $x = 500$ m past the observer?

(b) Graph $\frac{d\theta}{dx}$ as a function of x and determine the point at which θ changes most rapidly.

Observe that $\cot\theta = x/400$. So

$$\theta = \cot^{-1}\frac{x}{400}.$$

Taking the derivative, we have

$$\frac{d\theta}{dx} = \frac{-1}{1+(x/400)^2}\cdot\frac{1}{400} = \frac{-400}{400^2+x^2}.$$

(Note that $d\theta/dx$ does *not* depend on the speed of the plane, because we are looking at the change with respect to x, not with respect to time!)

In particular, when x is 500 meters, we get

$$\frac{d\theta}{dx} = \frac{-400}{400^2+500^2} \approx -0.000976,$$

which is in units of radians per meter.

The angle θ is changing the most rapidly when $|d\theta/dt|$ is as large as possible. By the chain rule, we have

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt} = \frac{d\theta}{dx}\cdot 70\,\text{m/s}.$$

We have

$$\left|\frac{d\theta}{dx}\right| = \frac{400}{400^2+x^2},$$

and this is as large as possible when the denominator $400^2 + x^2$ is as small as possible, i.e., when $x = 0$. In other words, the angle is changing fastest when the plane is directly overhead.