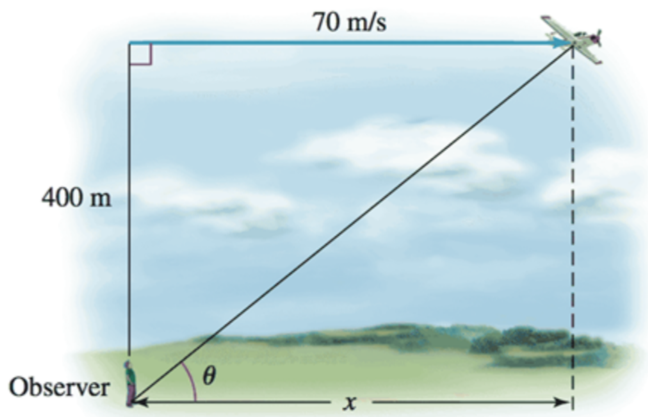
Dr. Daniel Hast, *drhast@bu.edu*

Today's topics

1 Derivatives of Inverse Trigonometric Functions

Example 1 (§3.10, Ex. 46)**.** A small plane, moving at 70 m/s, flies horizontally on a line 400 m directly above an observer. Let θ be the angle of elevation of the plane (see figure).



(a) What is the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when the plane is $x = 500$ m past the observer?

(b) Graph $\frac{d\theta}{dx}$ as a function of x and determine the point at which θ changes most rapidly.

Observe that $\cot\theta = x/400$. So

$$\theta = \cot^{-1}\frac{x}{400}.$$

Taking the derivative, we have

$$\frac{d\theta}{dx} = \frac{-1}{1+(x/400)^2}\cdot\frac{1}{400} = \frac{-400}{400^2+x^2}.$$

(Note that $d\theta/dx$ does *not* depend on the speed of the plane, because we are looking at the change with respect to x, not with respect to time!)

In particular, when x is 500 meters, we get

$$\frac{d\theta}{dx} = \frac{-400}{400^2+500^2} \approx -0.000976,$$

which is in units of radians per meter.

The angle θ is changing the most rapidly when $|d\theta/dt|$ is as large as possible. By the chain rule, we have

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt} = \frac{d\theta}{dx} \cdot 70\,\text{m/s}.$$

We have

$$\left|\frac{d\theta}{dx}\right| = \frac{400}{400^2 + x^2},$$

and this is as large as possible when the denominator $400^2 + x^2$ is as small as possible, i.e., when $x = 0$. In other words, the angle is changing fastest when the plane is directly overhead.

2 Related rates

Briggs–Cochran–Gillett §3.11, pp. 227–236

We revisit the idea of the derivative as the rate of change of a function, in the context of problems that have related variables changing with respect to time.

Here is a general outline of how to solve related rates problems:

(1) Read the problem carefully, and assign a variable name to each relevant quantity (both the quantity to be determined and those for which information is given). It often helps to draw a sketch of the situation.

(2) Write equations symbolically expressing the relations between the quantities.

(3) Differentiate equations with respect to time as appropriate.

(4) Substitute known/given values for any quantities that are given, and solve the resulting equations for the desired quantity.

(5) Check that your answer has the correct units and is physically reasonable (for example, that it has the correct sign and is not implausibly large or small).

Example 2 (§3.11, Ex. 12)**.** The sides of a square decrease in length at a rate of 1 m/s.

(a) At what rate is the area of the square changing when the sides are 5 m long?

(b) At what rate are the lengths of the diagonals of the square changing?

Solution. First, we label the relevant quantities: Let s be the side length (in meters), A be the area (in square meters), D be the length of the diagonal (in meters), and t be time (in seconds). These quantities are related by the following equations:

$$A = s^2, \qquad D = \sqrt{s^2 + s^2} = \sqrt{2} \cdot s.$$

Differentiating these equations with respect to time, we obtain

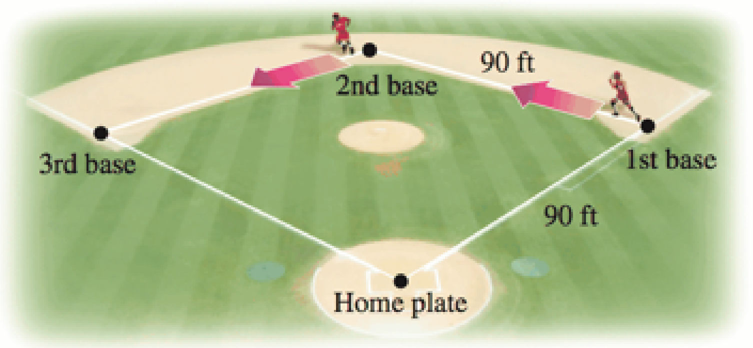
$$\frac{dA}{dt} = 2s\frac{ds}{dt}, \qquad \frac{dD}{dt} = \sqrt{2} \cdot \frac{ds}{dt}.$$

We are given that $ds/dt = -1\,\mathrm{m/s}$. So:

(a) The rate of change of the area is $dA/dt = -2s$ m/s. When $s = 5$ m, we obtain $dA/dt = -10\,\mathrm{m}^2/\mathrm{s}$. The units are correct: a rate of change of area should be in square meters per second. The sign is also correct: if the side length is decreasing, then the area is also decreasing.

(b) The rate of change of the lengths of the diagonals is $dD/dt = -\sqrt{2}\,\mathrm{m/s}$. The units are correct: a rate of change of a length is in meters per second. The sign is also correct: the diagonal length is decreasing.

Example 3 (§3.11, Ex. 28)**.** Runners stand at first and second base in a baseball game. At the moment a ball is hit, the runner at first base runs to second base at 18 ft/s; simultaneously the runner on second runs to third base at 20 ft/s. How fast is the distance between the runners changing 1 second after the ball is hit (see figure)? (Hint: the distance between consecutive bases is 90 ft and the bases lie at the corners of a square.)



Solution. Let x be the distance (in feet) of the first base runner from first base, y the distance of the second base runner from second base, and D the distance between the two runners. By the Pythagorean theorem and the fact that the bases are on a square with side length 90 feet, we have the relation

$$D^2 = (90 - x)^2 + y^2.$$

Differentiating, we obtain

$$2D\frac{dD}{dt} = -2(90 - x)\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

When $t = 1$ s, we have $x = 18$ ft and $y = 20$ ft, so

$$D = \sqrt{(90 - 18)^2 + 20^2} = \sqrt{5584}\,\text{ft}.$$

We are given $dx/dt = 18$ ft/s and $dy/dt = 20$ ft/s. Substituting into the above equation, we obtain

$$2\sqrt{5584}\frac{dD}{dt} = -2 \cdot (90 - 18) \cdot 18 + 2 \cdot 20 \cdot 20\,\text{ft}^2/\text{s}.$$

Solving for dD/dt yields

$$\frac{dD}{dt} = \frac{-2 \cdot (90 - 18) \cdot 18 + 2 \cdot 20 \cdot 20}{2\sqrt{5584}} = \frac{-1792}{2\sqrt{5584}} = \frac{-224}{\sqrt{349}} \approx -11.99\,\text{ft/s}.$$