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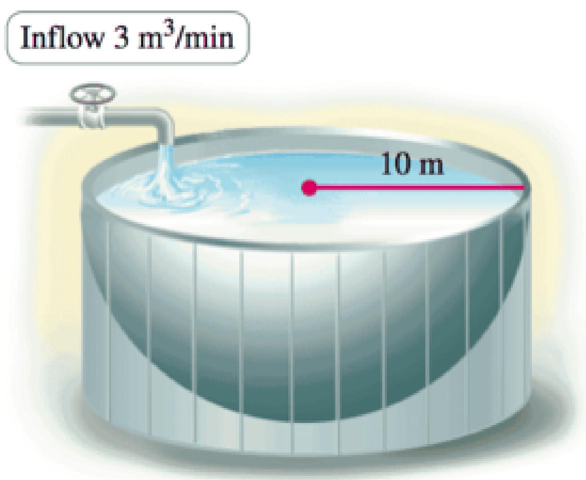
Today's topics

1 Related rates

Briggs–Cochran–Gillett §3.11, pp. 227–236

Example 1 (§3.11, Ex. 39)**.** A hemispherical tank with a radius of 10 m is filled from an inflow pipe at a rate of 3 $\mathrm{m}^3/\mathrm{min}$ (see figure). How fast is the water level rising when the water level is 5 m from the bottom of the tank? (Hint: the volume of a cap of thickness h sliced from a sphere of radius r is $\pi h^2(3r - h)/3$.)



Solution. Let h be the water level (measured at the center of the tank), and V be the volume of water in the tank. Then

$$V = \frac{1}{3}\pi h^2(30 - h) = 10\pi h^2 - \frac{1}{3}\pi h^3.$$

Differentiating, we get

$$\frac{dV}{dt} = 20\pi h\frac{dh}{dt} - \pi h^2\frac{dh}{dt} = (20h - h^2)\pi\frac{dh}{dt}.$$

At $h = 5$ m, we get

$$\frac{dV}{dt} = (75\pi\ \mathrm{m}^2)\frac{dh}{dt}.$$

We are given $dV/dt = 3\,\mathrm{m}^3/\mathrm{min}$, so

$$\frac{dh}{dt} = \frac{dV/dt}{75\pi\,\mathrm{m}^2} = \frac{3\,\mathrm{m}^3/\mathrm{min}}{75\pi\,\mathrm{m}^2} = \frac{1}{25\pi}\,\mathrm{m/min}.$$

The units are correct: the water level is in meters per minute. The sign is also correct: water is flowing into the tank, and the water level is rising.

Example 2 (§3.11, Ex. 42)**.** A 12-foot ladder is leaning against a vertical wall when Jack begins pulling the foot of the ladder away from the wall at a rate of 0.2 ft/s. What is the configuration of the ladder at the instant that the vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder?

Solution. Let x be the horizontal distance from the bottom of the ladder to the wall, and let y be the vertical distance from the ground to the top of the ladder. By the Pythagorean theorem,

$$x^2 + y^2 = 12^2\,\mathrm{ft}^2.$$

Differentiating, we obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

We are given $dx/dt = 0.2\,\mathrm{ft/s}$, and we are considering the time at which the vertical speed is equal to the horizontal speed, i.e., $dy/dt = -0.2\,\mathrm{ft/s}$. Substitute into the above equation:

$$0.4x - 0.4y = 0,$$

so $x = y$. Since $x^2 + y^2 = 12^2\,\mathrm{ft}^2$, this means $x = y = \sqrt{12^2/2} = 6\sqrt{2}\,\mathrm{ft}$.

Example 3 (§3.11, Ex. 48)**.** A jet ascends at a 10° angle from the horizontal with an airspeed of 550 mi/hr (its speed along its line of flight is 550 mi/hr). How fast is the altitude of the jet increasing? If the sun is directly overhead, how fast is the shadow of the jet moving on the ground?

Solution. Let x be the distance the shadow has traveled, h the altitude of the jet, and z the distance the jet itself has traveled (along its line of flight). Since the angle of ascent is 10°, we have

$$h = \sin(10^\circ) \cdot z \qquad \text{and} \qquad x = \cos(10^\circ) \cdot z.$$

Note that $\sin(10^\circ) \approx 0.17365$ and $\cos(10^\circ) \approx 0.985$. Also, we are given $dz/dt = 550\,\mathrm{mi/hr}$. Differentiating the above equations and substituting, we obtain

$$\begin{aligned}
\frac{dh}{dt} &= \sin(10^\circ) \cdot \frac{dz}{dt} \approx 0.17365 \cdot 550\,\mathrm{mi/hr} \approx 95.51\,\mathrm{mi/hr}, \\
\frac{dx}{dt} &= \cos(10^\circ) \cdot \frac{dz}{dt} \approx 0.98481 \cdot 550\,\mathrm{mi/hr} \approx 541.64\,\mathrm{mi/hr}.
\end{aligned}$$

So the altitude of the jet is increasing at about 95.51 miles per hour, while the shadow of the jet is moving at about 541.64 miles per hour on the ground.

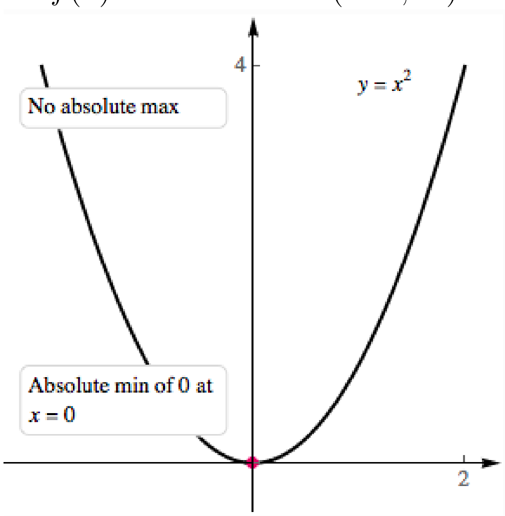
2 Maxima and minima

Briggs–Cochran–Gillett §4.1, pp. 241–250

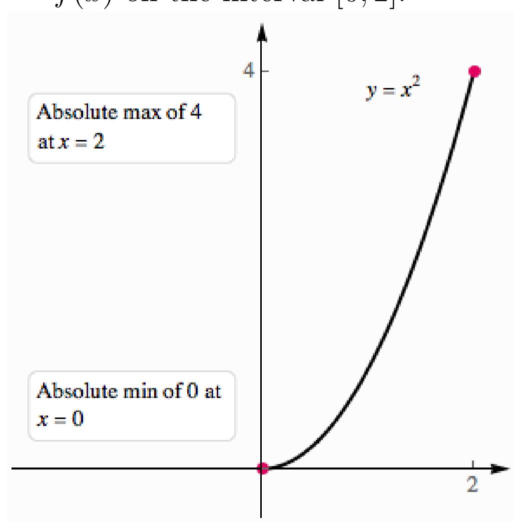
Definition 4 (Absolute Maximum and Minimum)**.** *Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an* ***absolute maximum*** *value of f on D. If $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an* ***absolute minimum*** *value of f on D. An* ***absolute extreme value*** *is either an absolute maximum or an absolute minimum value.*

The existence and location of absolute extreme values depend on both the *function* and the *interval* of interest. The figure below shows various cases for the function $f(x) = x^2$. Note that if the interval of interest is not closed, a function might not attain absolute extreme values.

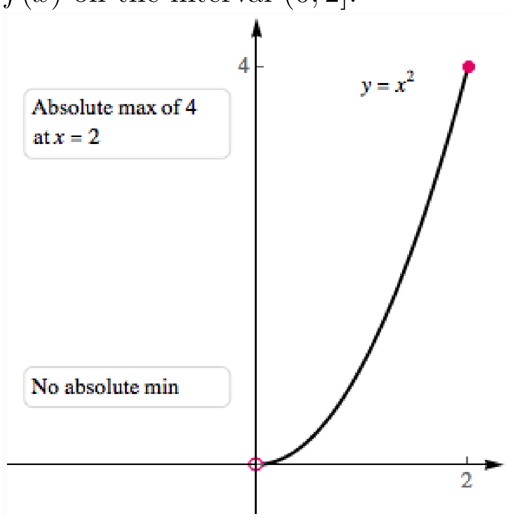
$f(x)$ on the interval $(-\infty, \infty)$:



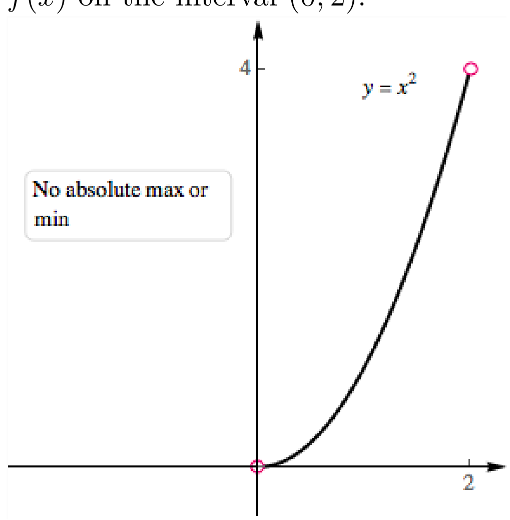
$f(x)$ on the interval $[0, 2]$:



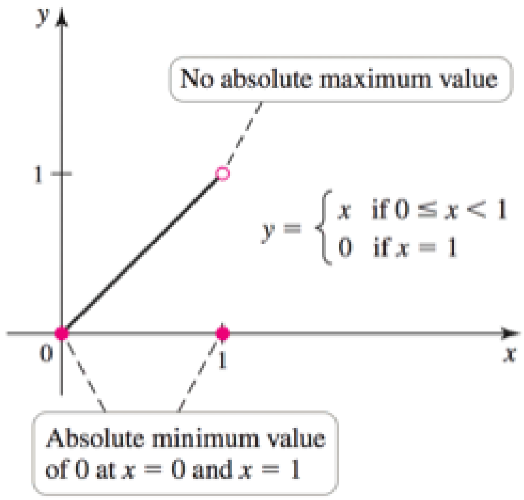
$f(x)$ on the interval $(0, 2]$:

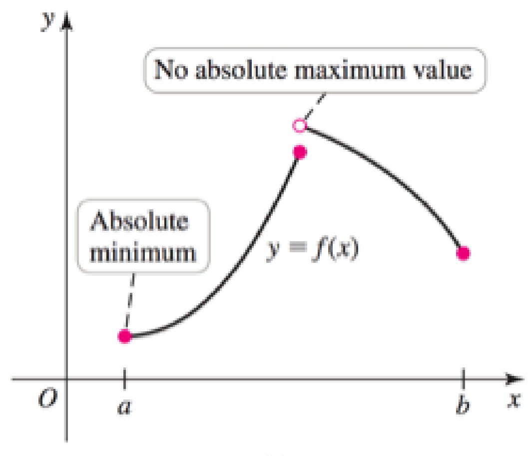


$f(x)$ on the interval $(0, 2)$:

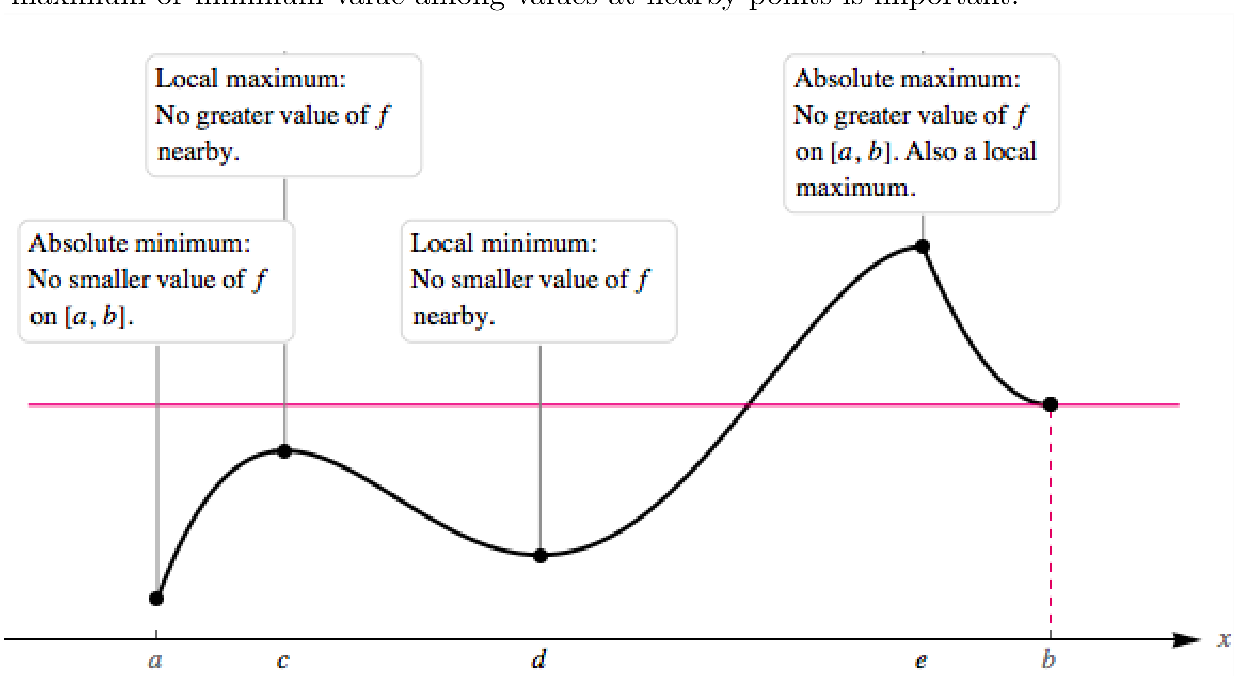


Note that defining a function on a closed interval is not enough to guarantee the existence of absolute extreme values. Both of the following functions are defined at every point of a closed interval, but neither function attains an absolute maximum—the discontinuity in each function prevents it from happening.





The function below is defined on the interval $[a, b]$. It has an absolute minimum at the endpoint a and an absolute maximum at the interior point e. In addition, the function has special behavior at c, where its value is greatest *among values at nearby points* and at d, where its value is least *among values at nearby points*. A point at which a function takes on the maximum or minimum value among values at nearby points is important.



Definition 5 (Local maximum and minimum values)**.** *Suppose c is an interior point of some interval I on which f is defined. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a* ***local***

maximum *value of* f*. If* $f(c) \leq f(x)$ *for all* x *in* I*, then* $f(c)$ *is a* ***local minimum*** *value of* f*.*

In this course, we take the convention that local maximum values and local minimum values occur only at ***interior points*** of the interval(s) of interest.