Dr. Daniel Hast, *drhast@bu.edu*

Today's topics

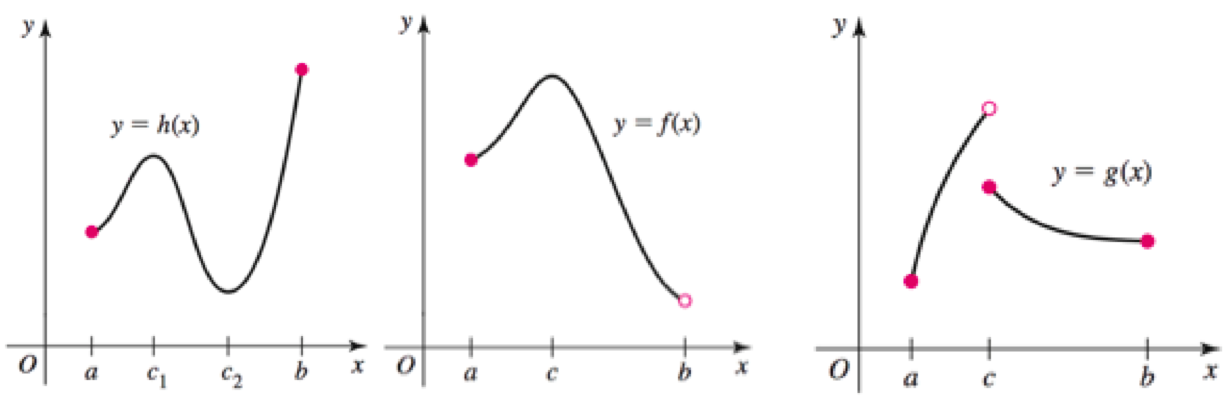
1 Maxima and minima

Briggs–Cochran–Gillett §4.1, pp. 241–250

Definition 1 (Local maximum and minimum values)**.** *Suppose c is an interior point of some interval I on which f is defined. If $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a **local maximum** value of f. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a **local minimum** value of f.*

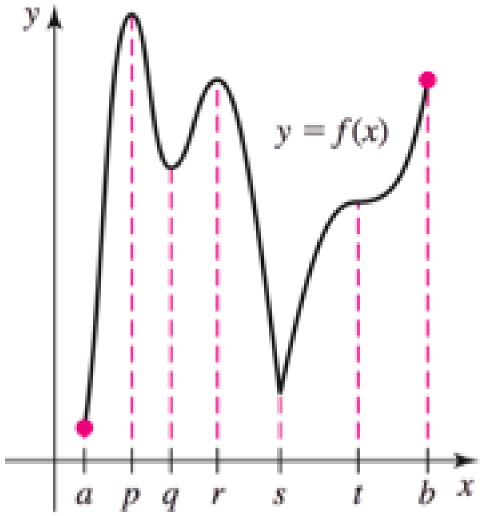
In this course, we take the convention that local maximum values and local minimum values occur only at ***interior points*** of the interval(s) of interest.

Example 2 (§4.1, Ex. 11, 12, 13)**.** Use the following graphs to identify the points (if any) on the interval $[a, b]$ at which the function has absolute or local extreme values.



Remark 3. A function can have an absolute maximum or minimum at more than one point (if the maximum or minimum value occurs more than once). An extreme case is a constant function: *every* interior point of a constant function is both an absolute maximum and an absolute minimum. Also, note that a local maximum doesn't need to be strictly greater than all nearby values, just greater than *or equal to* all nearby values.

Example 4 (§4.1, Ex. 16)**.** Identify the points on the interval $[a, b]$ at which local and absolute extreme values occur.



Example 5 (§4.1, Ex. 22)**.** Sketch a graph of a function f continuous on $[0, 4]$ satisfying the following properties: $f'(x) = 0$ at $x = 1$ and $x = 3$; $f'(2)$ is undefined; f has an absolute maximum at $x = 2$; f has neither a local maximum nor a local minimum at $x = 1$; and f has an absolute minimum at $x = 3$.

Solution. Each person might come up with a different function for the above problem, but here's the strategy I typically use for this sort of problem:

1. Place a dot at each x-value where there are properties needing to be satisfied, making sure to choose the largest (respectively, smallest) y-values for an absolute maximum (resp., absolute minimum).

2. Draw a small part of the graph around each dot where we need to satisfy conditions on the derivative—for example, a small horizontal line if the derivative is zero, or a corner if the derivative is undefined.

3. Connect the dots in a way that satisfies all the properties.

(In practice, this sort of method tends to be easier than trying to draw the whole graph freehand all at once.)

1.1 Extreme values and critical points

It turns out that two conditions ensure the existence of absolute maximum and minimum values on an interval: the function must be continuous on the interval, and the interval must be closed and bounded:

Theorem 6 (Extreme Value Theorem)**.** *A function that is continuous on a closed interval* $[a, b]$ *has an absolute maximum value and an absolute minimum value on that interval.*

It turns out that local maxima and minima occur at points in the open interval (a, b) where the derivative is zero and at points where the derivative fails to exist. We now make this observation precise.

Theorem 7 (Local Extreme Value Theorem)**.** *If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$.*

Local extrema can also occur at points c where $f'(c)$ does not exist. Because local extrema may occur at points c where $f'(c) = 0$ and where $f'(c)$ does not exist, we make the following definition.

Definition 8 (Critical point)**.** *An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a critical point of f*

Example 9 (§4.1, Ex. 25, 34, 36)**.** Find the critical points of the following functions on the domain or on the given interval. Use a graphing utility to determine whether each critical point corresponds to a local maximum, local minimum, or neither.

1. $f(x) = \dfrac{x^3}{3} - 9x$ on $[-7, 7]$
2. $f(x) = \sin x \cos x$ on $[0, 2\pi]$
3. $f(x) = x^2 - 2\ln(x^2 + 1)$
4. $f(x) = x^{2/3}$
5. $f(x) = |x - 2|$

Solution.

1. $f'(x) = x^2 - 9$, so the critical points are at -3 and 3.
2. $f'(x) = \cos^2 x - \sin^2 x$, so the critical points are where $\sin x = \pm\cos x$, that is, at $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.
3. $f'(x) = 2x - \dfrac{4x}{x^2 + 1}$, so the critical points are where
$$2x - \frac{4x}{x^2 + 1} = 0.$$
Multiplying both sides by $x^2 + 1$ (which is always nonzero), we obtain
$$2x(x^2 + 1) - 4x = 0.$$
We have $2x(x^2 + 1) - 4x = 2x(x^2 - 1) = 2x(x - 1)(x + 1)$, so the critical points are at $x = 0, -1, 1$.
4. $f'(x) = \frac{2}{3}x^{-1/3}$, which is never zero, but is undefined at $x = 0$, which is the one critical point.
5. Note that
$$|x - 2| = \begin{cases} x - 2 & \text{if } x \geq 2, \\ 2 - x & \text{if } x < 2. \end{cases}$$
So $f'(x) = 1$ if $x > 2$, $f'(x) = -1$ if $x < 2$, and $f'(2)$ is undefined, so there is a critical point at 2.

1.2 Locating Absolute Maxima and Minima

If we have a continuous function on a closed interval **how do we locate its maximum and minimum values**?

Assume the function f is continuous on the closed interval $[a, b]$.

1. Locate the critical points c_i in (a, b).
2. Evaluate f
   - at the critical points $x = c_i$
   - and at the endpoints $x = a$ and $x = b$.
3. Choose the largest and smallest values of f from Step 2.