Dr. Daniel Hast, *drhast@bu.edu*

Today's topics

1 Maxima and minima

Briggs–Cochran–Gillett §4.1, pp. 241–250

Example 1. Find the critical points of the functions below on the given intervals. Determine the absolute extreme values of f on the given interval when they exist. Use a graphing utility to confirm your conclusions.

1. $f(x) = \dfrac{x}{(x^2+3)^2}$ on $[-2, 2]$
2. $f(x) = \left|2x - x^2\right|$ on $[-2, 3]$

Solution. 1. We compute the derivative:

$$f'(x) = \frac{(x^2+3)^2 - x \cdot 2(x^2+3) \cdot 2x}{(x^2+3)^4} = \frac{(x^2+3) - 4x^2}{(x^2+3)^3} = \frac{-3x^2+3}{(x^2+3)^3}.$$

Since $x^2 + 3$ is always positive, this is defined everywhere on $[-2, 2]$, so the critical points are exactly where $f'(x) = 0$. Setting the numerator equal to zero, we need to solve $-3x^2 + 3 = 0$, or equivalently, $x^2 = 1$. So the critical points are at $x = \pm 1$.

To determine the absolute extreme values, we need to evaluate f at the critical points and at the endpoints:

$$f(-2) = \frac{-2}{49}, \qquad f(-1) = \frac{-1}{16}, \qquad f(1) = \frac{1}{16}, \qquad f(2) = \frac{2}{49}.$$

Since $1/16 > 2/49$, the absolute maximum is $1/16$ (at $x = 1$) and the absolute minimum is $-1/16$ (at $x = -1$).

2. Observe that $2x - x^2 = x(2 - x)$ is negative when $x < 0$, positive when $0 < x < 2$, and negative when $2 < x$. So

$$f'(x) = \begin{cases} 2 - 2x & \text{if } 0 < x < 2, \\ 2x - 2 & \text{if } x < 0 \text{ or } 2 < x. \end{cases}$$

The derivative is undefined at 0 and 2. Also, $f'(x) = 0$ if and only if $x = 1$. So the critical points are 0, 1, and 2. Evaluating the function at critical points and endpoints:

$$f(-2) = 8, \qquad f(0) = 0, \qquad f(1) = 1, \qquad f(2) = 0, \qquad f(3) = 3.$$

So the absolute maximum value is 8 (at $x = -2$), and the absolute minimum value is 0 (at both $x = 0$ and $x = 2$).

In the following example, we locate absolute max/min on an open interval by inspection of the graph of the function.

Example 2. All rectangles with an area of $A > 0$ have a perimeter given by $P(x) = 2x+2A/x$, where x is the length of one side of the rectangle. Find the absolute minimum value for the perimeter function on the interval $(0, \infty)$. What are the dimensions of the rectangle with minimum perimeter?

Solution. The derivative of the perimeter function is

$$P'(x) = 2 - \frac{2A}{x^2}.$$

This is defined everywhere on $(0, \infty)$, so the critical points are exactly where $P'(x) = 0$, that is, where $2 = 2A/x^2$. This is equivalent to $x^2 = A$, which has solutions $x = \sqrt{A}$ and $x = -\sqrt{A}$. However, the domain we are considering is $(0, \infty)$, so we discard the negative solution and are left with $x = \sqrt{A}$.

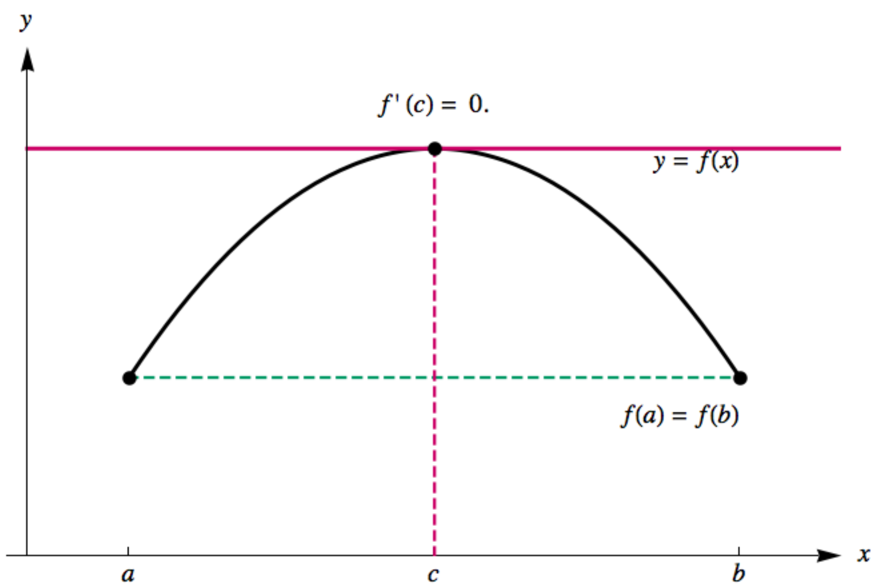
The perimeter at $x = \sqrt{A}$ is $P(\sqrt{A}) = 2\sqrt{A} + 2A/\sqrt{A} = 4\sqrt{A}$. In other words, we've verified that the rectangle of area A with the smallest perimeter is a square.

2 Mean Value Theorem

Briggs–Cochran–Gillett §4.2, pp. 250–257

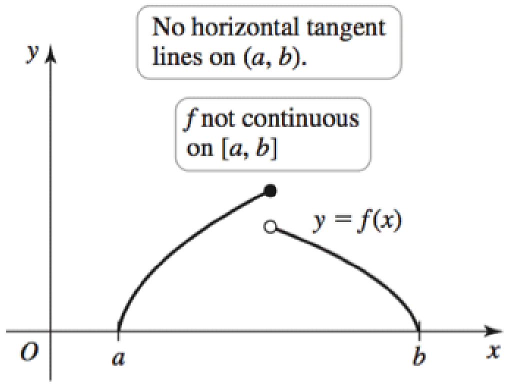
Today we will look at one of the central results in calculus: the Mean Value Theorem (MVT). Some of the theorems we have already seen about derivatives are a consequence of this theorem. We start by looking at a preliminary result, which is a particular case of the MVT: Rolle's theorem.

2.1 Rolle's Theorem

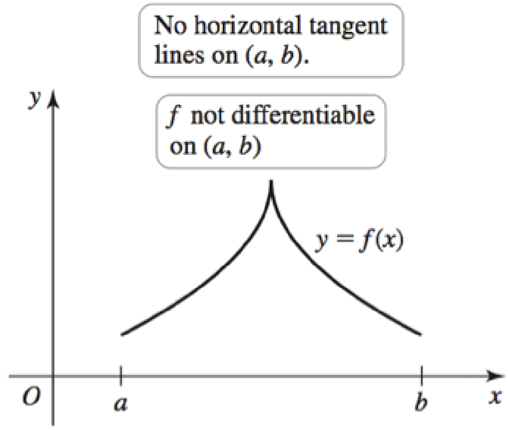


Theorem 3 (Rolle's Theorem)**.** *Let f be a continuous function on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.*

The continuity and differentiability conditions are essential, otherwise the statement is not true!

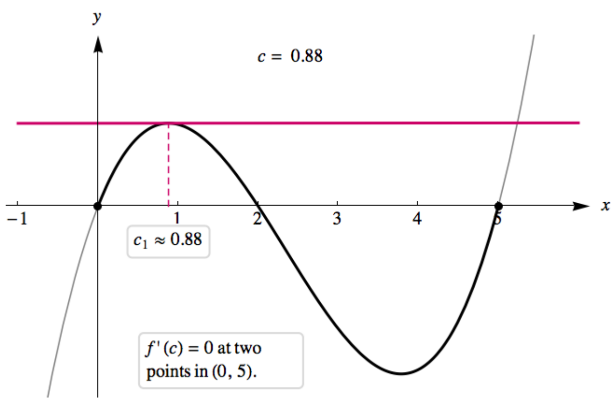


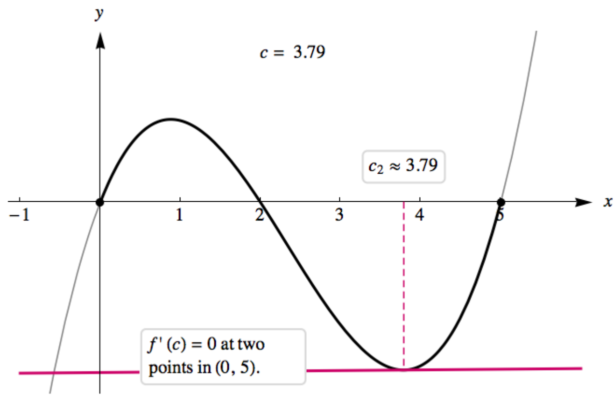
(a)



(b)

Also note that there might be more than one c that satisfies the theorem:





Example 4 (§4.2, Ex. 16)**.** Determine whether Rolle's theorem applies to the function $f(x) = x^3 - 2x^2 - 8x$ on the interval $[-2, 4]$. If so, find the point(s) that are guaranteed to exist by Rolle's Theorem.

Solution. Since f is a polynomial function, f is continuous and differentiable everywhere. We also have $f(-2) = 0$ and $f(4) = 0$, so $f(-2) = f(4)$. Thus Rolle's theorem does apply to f.

We want to find points x in $(-2, 4)$ such that $f'(x) = 0$. We have

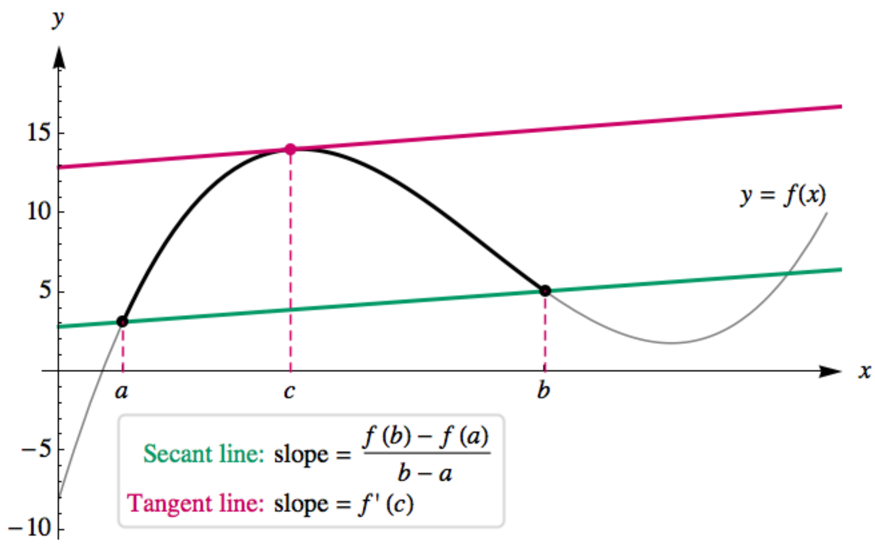
$$f'(x) = 3x^2 - 4x - 8.$$

By the quadratic formula, the roots of this quadratic function are

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 3 \cdot (-8)}}{2 \cdot 3} = \frac{4 \pm 4\sqrt{7}}{6} = \frac{2 \pm 2\sqrt{7}}{3}.$$

These are both in the interval $(-2, 4)$ (they are approximately -1.1 and 2.4), so these are the points guaranteed to exist by Rolle's theorem.

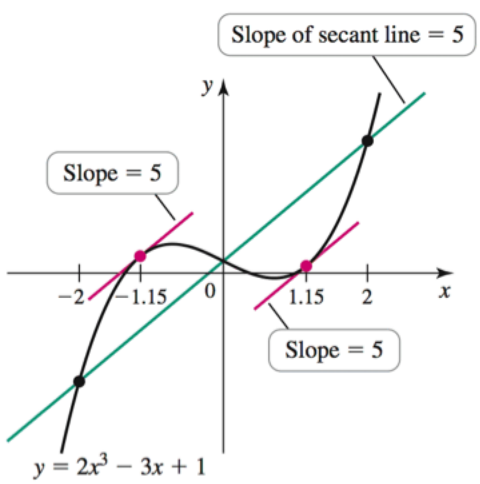
2.2 The Mean Value Theorem (MVT)'s statement



Theorem 5 (Mean Value Theorem)**.** *Let f be a continuous function on a closed interval $[a, b]$ and differentiable on (a, b). Then there is at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note that, just like in the case of Rolle's Theorem, there might be more than one c that satisfies the MVT:



Example 6 (§4.2, Ex. 26, 31)**.**

(a) Determine whether the MVT applies to the following functions on the given interval $[a, b]$. Explain why or why not.

(b) If so, find the point(s) that are guaranteed to exist by the MVT and sketch the function and the line that passes through $(a, f(a))$ and $(b, f(b))$. Mark the points P at which the slope of the function equals the slope of the secant line and sketch the tangent line at P.

1. $f(x) = \ln(2x)$, $[1, e]$
2. $f(x) = 2x^{1/3}$, $[-8, 8]$

Solution. 1. (a) The function $f(x) = \ln(2x)$ is continuous and differentiable on $(0, \infty)$, so the MVT applies.

(b) The MVT says there exists c in $(1, e)$ such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1} = \frac{\ln(2e) - \ln(2)}{e - 1} = \frac{\ln(2) + \ln(e) - \ln(2)}{e - 1} = \frac{1}{e - 1}.$$

For all c in $(0, \infty)$, we have

$$f'(c) = \frac{2}{2c} = \frac{1}{c}.$$

So we want to solve the equation

$$\frac{1}{c} = \frac{1}{e - 1}.$$

This has the solution $c = e - 1$, which is indeed in the interval $(1, e)$.

2. The function $f(x) = 2x^{1/3}$ is not differentiable at $x = 0$, so the MVT does not apply. Let's try to solve for points with that property anyway:

$$f'(x) = \frac{2}{3x^{2/3}},$$

and we have $f(-8) = -4$ and $f(8) = 4$, so we're looking for points c such that

$$f'(c) = \frac{f(8) - f(-8)}{8 - (-8)} = \frac{4 + 4}{8 + 8} = \frac{1}{2}.$$

In other words, we want $\frac{2}{3}x^{-2/3} = \frac{1}{2}$, or equivalently,

$$x^{2/3} = \frac{4}{3},$$

which has solutions $x = \pm\frac{8}{\sqrt{27}}$. So, in this particular case, points satisfying the conclusion of the MVT exist anyway; however, this is not a consequence of the MVT, since the differentiability hypothesis is not satisfied.