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Today's topics

1 Consequences of the MVT

Briggs–Cochran–Gillett §4.2, pp. 250–257

Theorem 1 (Consequences of the MVT)**.**

1. ***Zero derivative implies constant function***
   If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

2. ***Functions with equal derivatives differ by a constant***
   If two functions have the property that $f'(x) = g'(x)$, for all x of an interval I, then f and g differ by a constant on the interval, i.e., $f(x) - g(x) = constant$ on I.

3. ***Intervals of increase and decrease***
   Suppose f is continuous on I and differentiable on all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

Example 2 (§4.2, Ex. 36)**.** Without evaluating derivatives, which of the functions

$$f(x) = \ln x, \quad g(x) = \ln 2x, \quad h(x) = \ln(x^2), \text{ and } p(x) = \ln(10x^2)$$

have the same derivative?

Solution. Using properties of logarithms,

$$g(x) = \ln 2 + \ln x = f(x) + \ln 2$$

and

$$p(x) = \ln 10 + \ln(x^2) = h(x) + \ln 10.$$

So $f' = g'$ and $h' = p'$. On the other hand,

$$h(x) = 2\ln x = 2f(x),$$

so f and h do *not* differ by a constant function, so $f' \neq h'$.

Example 3 (§4.2, Ex. 38)**.** Find all functions f whose derivative is $f'(x) = x + 1$.

Solution. One such function is $f(x) = \frac{1}{2}x^2 + x$. Since functions with equal derivatives differ by a constant, all such functions are of the form

$$f(x) = \frac{1}{2}x^2 + x + C$$

for some real number C.

2 What derivatives tell us

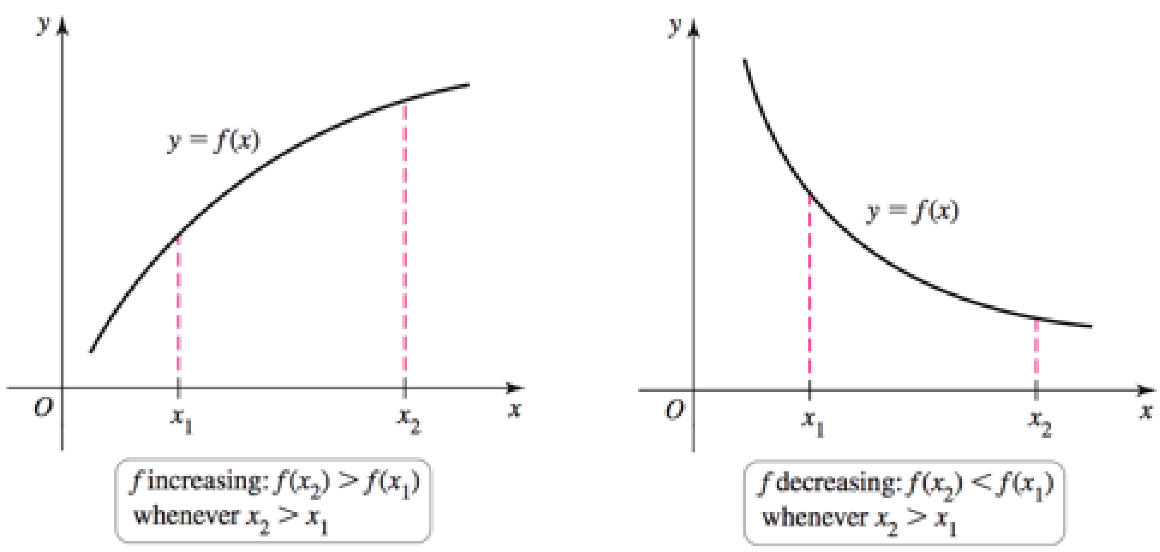
Briggs–Cochran–Gillett §4.3, pp. 257–262

Earlier this week, we saw that the derivative helps us find critical points of functions. Today we explore how the derivative can tell us more about the behavior of functions.

2.1 Increasing and decreasing functions

Definition 4 (Increasing and decreasing functions)**.** *Suppose a function f is defined on an interval I. We say that:*

- *f is* ***increasing*** *on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.*
- *f is* ***decreasing*** *on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.*



Theorem 5 (Test for intervals of increase and decrease)**.** *Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.*

Example 6 (§4.3, Ex. 20, 30, 36)**.** For each of the following functions, find the intervals on which f is increasing and decreasing.

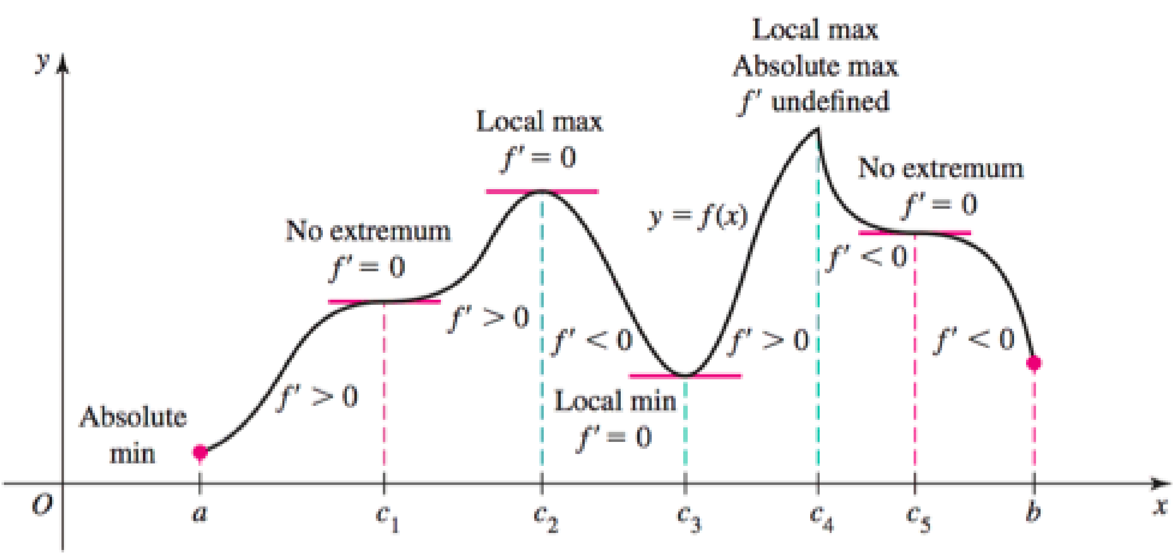
1. $f(x) = x^2 - 16$
2. $f(x) = \frac{e^x}{e^{2x}+1}$
3. $f(x) = x^2\sqrt{9-x^2}$ on $(-3, 3)$

Solution.

1. We have $f'(x) = 2x$, which is positive when $x > 0$ and negative when $x < 0$. So f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.
2. We have
$$f'(x) = \frac{e^x(e^{2x}+1) - e^x \cdot 2e^{2x}}{(e^{2x}+1)^2} = \frac{e^x(1-e^{2x})}{(e^{2x}+1)^2}.$$
Since e^x and $e^{2x}+1$ are always positive, the sign of $f'(x)$ is the same as the sign of $1 - e^{2x}$, which is positive on $(-\infty, 0)$ and negative on $(0, \infty)$. So f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.
3. We compute the derivative
$$f'(x) = 2x\sqrt{9-x^2} + x^2 \cdot \frac{1}{2\sqrt{9-x^2}} \cdot (-2x) = 2x\sqrt{9-x^2} - \frac{x^3}{\sqrt{9-x^2}}.$$
Multiplying by a positive number does not change the sign, and $\sqrt{9-x^2}$ is positive on $(-3, 3)$, so the following function has the same sign as $f'(x)$:
$$f'(x)\sqrt{9-x^2} = 2x(9-x^2) - x^3 = 18x - 3x^3 = 3x(6-x^2).$$
So f is increasing on $(-\infty, -\sqrt{6})$, decreasing on $(-\sqrt{6}, 0)$, increasing on $(0, \sqrt{6})$, and decreasing on $(\sqrt{6}, \infty)$.

2.2 Identifying local maxima and minima: The first derivative test

The figure below shows typical features of a function on an interval $[a, b]$.



At local maxima or minima (c_2, c_3, c_4), the derivative f' changes sign. Although c_1 and c_5 are critical points, the sign of f' is the same on both sides near these points, so there is no local maximum or minimum at these points. Note: *critical points do not always correspond to local extreme values*!

The observations used to interpret the figure above are summarized in a very useful test for identifying local maxima and minima:

Theorem 7 (First derivative test)**.** *Suppose that f is continuous on an interval that contains a critical point c and assume f is differentiable on an interval containing c, except perhaps at c itself.*

- *If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.*
- *If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.*
- *If f' is positive on both sides near c or negative on both sides near c, then f has no local extreme value at c.*

Example 8 (§4.3, Ex. 46, 51, 52)**.** For each of the following functions:

(a) Locate the critical points of f.

(b) Use the First Derivative Test to locate the local maximum and minimum values.

(c) Identify the absolute maximum and minimum values of the function on the given interval (when they exist).

1. $f(x) = -x^2 - x + 2$ on $[-4, 4]$
2. $f(x) = x^{2/3}(x - 5)$ on $[-5, 5]$
3. $f(x) = \frac{x^2}{x^2-1}$ on $[-4, 4]$

Solution. 1. We have $f'(x) = -2x-1$, so f has a critical point at $-1/2$. Since f' changes sign from positive to negative as x increases through this critical point, f has a local maximum at $-1/2$. Evaluating f at the critical point $-1/2$ and the endpoints $-4, 4$, we see that $f(-1/2) = 9/4$ is the absolute maximum value, and $f(4) = -18$ is the absolute minimum value.

2. We compute the derivative

$$f'(x) = \frac{2}{3}x^{-1/3}(x-5) + x^{2/3} = \frac{2x-10}{3x^{1/3}} + \frac{3x}{3x^{1/3}} = \frac{5x-10}{3x^{1/3}}.$$

This is zero exactly at $x = 2$, and is undefined at $x = 0$, so the critical points of f are at 0 and 2. Since $f'(x)$ is positive on $(-\infty, 0)$, negative on $(0, 2)$, and positive on $(2, \infty)$, the first derivative test tells us that f has a local maximum at 0 and a local minimum at 2. To find absolute extreme values, we evaluate f at the critical points and endpoints:

$$f(-5) = -10 \cdot 5^{2/3} \approx -34, \qquad f(0) = 0, \qquad f(2) = -3 \cdot 2^{2/3} \approx -4.8, \qquad f(5) = 0.$$

So the absolute minimum value is $f(-5) = -10 \cdot 5^{2/3}$, and the absolute maximum value is 0 and occurs at both $x = 0$ and $x = 5$.

3. The derivative is

$$f'(x) = \frac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}.$$

This is zero at $x = 0$, and is undefined at $x = \pm 1$. Note that f itself is undefined at $x = \pm 1$, so only $x = 0$ is a critical point.

We have $f'(x) > 0$ on the interval $(-1, 0)$, and $f'(x) < 0$ on the interval $(0, 1)$, so f has a local maximum at 0 by the first derivative test. As for absolute maxima and minima, there are none on $[-4, 4]$, since $\lim_{x \to 1^-} f(x) = -\infty$ and $\lim_{x \to 1^+} f(x) = \infty$.