Dr. Daniel Hast, *drhast@bu.edu*

Today's topics

1 What derivatives tell us

Briggs–Cochran–Gillett §4.3, pp. 257–270

1.1 The first derivative test, continued

Example 1 (§4.3, Ex. 46, 51, 52)**.** For each of the following functions:

(a) Locate the critical points of f.

(b) Use the First Derivative Test to locate the local maximum and minimum values.

(c) Identify the absolute maximum and minimum values of the function on the given interval (when they exist).

1. $f(x) = -x^2 - x + 2$ on $[-4, 4]$
2. $f(x) = x^{2/3}(x-5)$ on $[-5, 5]$
3. $f(x) = \frac{x^2}{x^2-1}$ on $[-4, 4]$

Solution. 1. We have $f'(x) = -2x - 1$, so f has a critical point at $-1/2$. Since f' changes sign from positive to negative as x increases through this critical point, f has a local maximum at $-1/2$. Evaluating f at the critical point $-1/2$ and the endpoints $-4, 4$, we see that $f(-1/2) = 9/4$ is the absolute maximum value, and $f(4) = -18$ is the absolute minimum value.

2. We compute the derivative

$$f'(x) = \frac{2}{3}x^{-1/3}(x-5) + x^{2/3} = \frac{2x-10}{3x^{1/3}} + \frac{3x}{3x^{1/3}} = \frac{5x-10}{3x^{1/3}}.$$

This is zero exactly at $x = 2$, and is undefined at $x = 0$, so the critical points of f are at 0 and 2. Since $f'(x)$ is positive on $(-\infty, 0)$, negative on $(0, 2)$, and positive on $(2, \infty)$, the first derivative test tells us that f has a local maximum at 0 and a local minimum at 2. To find absolute extreme values, we evaluate f at the critical points and endpoints:

$$f(-5) = -10 \cdot 5^{2/3} \approx -34, \qquad f(0) = 0, \qquad f(2) = -3 \cdot 2^{2/3} \approx -4.8, \qquad f(5) = 0.$$

So the absolute minimum value is $f(-5) = -10 \cdot 5^{2/3}$, and the absolute maximum value is 0 and occurs at both $x = 0$ and $x = 5$.

3. The derivative is
$$f'(x) = \frac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}.$$
This is zero at $x = 0$, and is undefined at $x = \pm 1$. Note that f itself is undefined at $x = \pm 1$, so only $x = 0$ is a critical point.

   We have $f'(x) > 0$ on the interval $(-1, 0)$, and $f'(x) < 0$ on the interval $(0, 1)$, so f has a local maximum at 0 by the first derivative test. As for absolute maxima and minima, there are none on $[-4, 4]$, since $\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$.

Recall that the Extreme Value Theorem guarantees absolute extreme values of continuous functions on closed intervals. What can be said about absolute extrema on intervals that are not closed? The following theorem gives a helpful test:

Theorem 2 (One local extremum implies absolute extremum)**.** *Suppose f is continuous on an interval I that contains exactly one local extremum at c.*

- *If a local maximum occurs at c, then $f(c)$ is the absolute maximum of f on I.*
- *If a local minimum occurs at c, then $f(c)$ is the absolute minimum of f on I.*

Example 3 (§4.3, Ex. 56, 58)**.** Verify that the following functions satisfy the conditions of Theorem 2 on their domains. Then find the location and value of the absolute extremum guaranteed by the theorem.

1. $f(x) = 4x + \dfrac{1}{\sqrt{x}}$
2. $f(x) = x\sqrt{3-x}$

Solution.

1. The domain of f is $(0, \infty)$. We have $f'(x) = 4 - \frac{1}{2}x^{-3/2}$, so $f'(x) = 0$ if and only if $x^{3/2} = 1/8$, which has unique positive solution $x = \frac{1}{4}$. Since $f'(x) < 0$ on $(0, \frac{1}{4})$ and $f'(x) > 0$ on $(\frac{1}{4}, \infty)$, the first derivative test implies f has a local minimum at $\frac{1}{4}$. This is the only local extremum, so by Theorem 2, f has an absolute minimum at $\frac{1}{4}$, with value $f(\frac{1}{4}) = 1 + \frac{1}{\sqrt{1/4}} = 3$.

2. The domain of f is $(-\infty, 3]$. We have
$$f'(x) = \sqrt{3-x} + x \cdot \frac{1}{2\sqrt{3-x}} \cdot (-1) = \frac{2(3-x)}{2\sqrt{3-x}} - \frac{x}{2\sqrt{3-x}} = \frac{6-3x}{\sqrt{3-x}},$$
so $f'(x) = 0$ if and only if $x = 2$. Since $f'(x) > 0$ on $(-\infty, 2)$ and $f'(x) < 0$ on $(2, 3)$, the first derivative test implies f has a local maximum at 2. This is the only local extremum, so by Theorem 2, f has an absolute maximum at 2, with value $f(2) = 2\sqrt{3-2} = 2$.

1.2 Concavity and inflection points

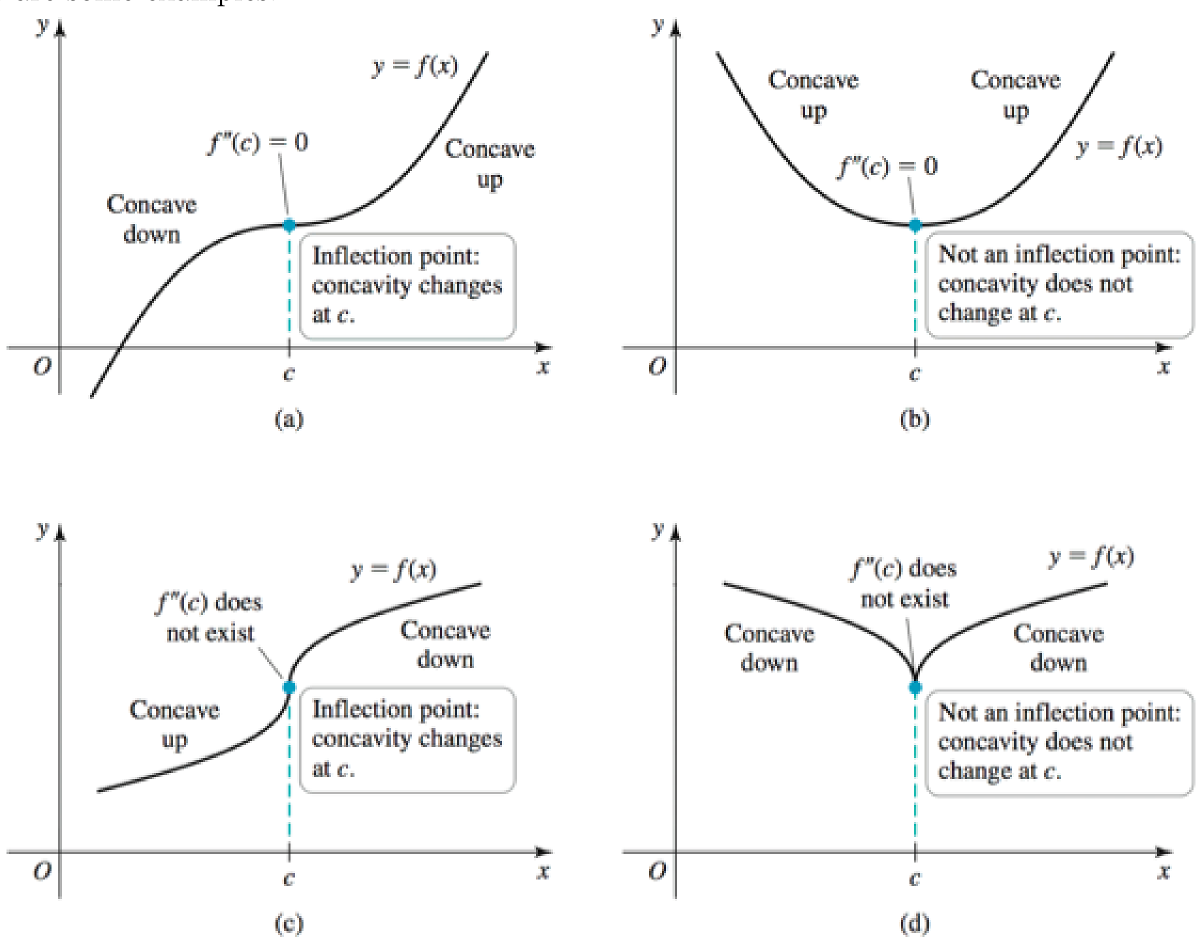
Just as the first derivative is related to the slope of tangent lines, the second derivative also has geometric meaning.

Definition 4 (Concavity and inflection point)**.** *Let f be differentiable on an open interval I. If f' is increasing on I, then f is* ***concave up*** *on I. If f' is decreasing on I, then f is* ***concave down*** *on I. If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an* ***inflection point*** *at c.*

Theorem 5 (Test for concavity)**.** *Suppose that f'' exists on an open interval I.*

- *If $f'' > 0$ on I, then f is concave up on I.*
- *If $f'' < 0$ on I, then f is concave down on I.*
- *If c is a point of I at which f'' changes sign at c (from positive to negative or negative to positive), then f has an inflection point at c.*

Here are some examples:



Example 6 (§4.3, Ex. 64, 68, 74)**.** Determine the intervals on which the following functions are concave up or concave down. Identify any inflection points.

1. $f(x) = -x^4 - 2x^3 + 12x^2$
2. $f(x) = 2x^2 \ln x - 5x^2$
3. $h(t) = 2 + \cos(2t)$ for $0 \le t \le \pi$.

Solution. 1. The derivative is $f'(x) = -4x^3 - 6x^2 + 24x$, and we want to find where f' is increasing and where it is decreasing. To do this, we compute the second derivative: $f''(x) = -12x^2 - 12x + 24 = -12(x+2)(x-1)$. This splits up the domain into three intervals:

- On $(-\infty, -2)$: f'' is negative, so f' is decreasing and f is concave down.
- On $(-2, 1)$: f'' is positive, so f' is increasing and f is concave up.
- On $(1, \infty)$: f'' is negative, so f' is decreasing and f is concave down.

Thus, the inflection points of f occur at -2 and 1.

2. The domain is $(0, \infty)$. We have

$$f'(x) = 4x \ln x + \frac{2x^2}{x} - 10x = 4x \ln x - 8x,$$
$$f''(x) = 4 \ln x + \frac{4x}{x} - 8 = 4 \ln x - 4 = 4(\ln x - 1).$$

So f'' is negative (and f' is decreasing, and f is concave down) on $(0, 1)$, and f'' is positive (and f' is increasing, and f is concave up) on $(1, \infty)$. Thus 1 is the only inflection point of f.

3. We have $h'(t) = -2\sin(2t)$ and $h''(t) = -4\cos(2t)$. The second derivative is negative (so f is concave down) on $(0, \pi/4)$ and $(3\pi/4, \pi)$, and positive (so f is concave up) on $(\pi/4, 3\pi/4)$; the inflection points are at $\pi/4$ and $3\pi/4$.

1.3 Second Derivative Test

Theorem 7 (Second Derivative Test for local extrema)**.** *Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.*

- *If $f''(c) > 0$, then f has a local minimum at c.*
- *If $f''(c) < 0$, then f has a local maximum at c.*
- *If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.*

Example 8 (§4.3, Ex. 78, 84)**.** Locate the critical points of the following functions. Then use the Second Derivative Test to determine (if possible) whether they correspond to local maxima or local minima.

1. $f(x) = 6x^2 - x^3$

2. $p(x) = \dfrac{e^x}{x+1}$

Solution. 1. The first derivative is $f'(x) = 12x - 3x^2 = 3x(4-x)$, so the critical points are at 0 and 4. The second derivative is $f''(x) = 12-6x$, so $f''(0) = 12$ and $f''(4) = -12$, so f has a local minimum at 0 and a local maximum at 4.

2. We have

$$p'(x) = \frac{e^x(x+1) - e^x}{(x+1)^2} = \frac{xe^x}{(x+1)^2},$$
$$p''(x) = \frac{(e^x + xe^x)(x+1)^2 - 2xe^x(x+1)}{(x+1)^4} = \frac{(e^x + xe^x)(x+1) - 2xe^x}{(x+1)^3} = \frac{e^x(x^2+1)}{(x+1)^3}.$$

So the only critical point of f is at 0. (Note that f is not defined at -1, so this is not a critical point.) Since $p''(0) = 1$, this is a local minimum.

Here is a recap of derivative properties: