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Today's topics

1 Derivative tests

Briggs–Cochran–Gillett §4.3, pp. 257–270

Theorem 1 (Second Derivative Test for local extrema)**.** *Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.*

- *If $f''(c) > 0$, then f has a local minimum at c.*
- *If $f''(c) < 0$, then f has a local maximum at c.*
- *If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, local minimum, or neither at c.*

Example 2 (§4.3, Ex. 78, 84)**.** Locate the critical points of the following functions. Then use the Second Derivative Test to determine (if possible) whether they correspond to local maxima or local minima.

1. $f(x) = 6x^2 - x^3$
2. $p(x) = \dfrac{e^x}{x+1}$

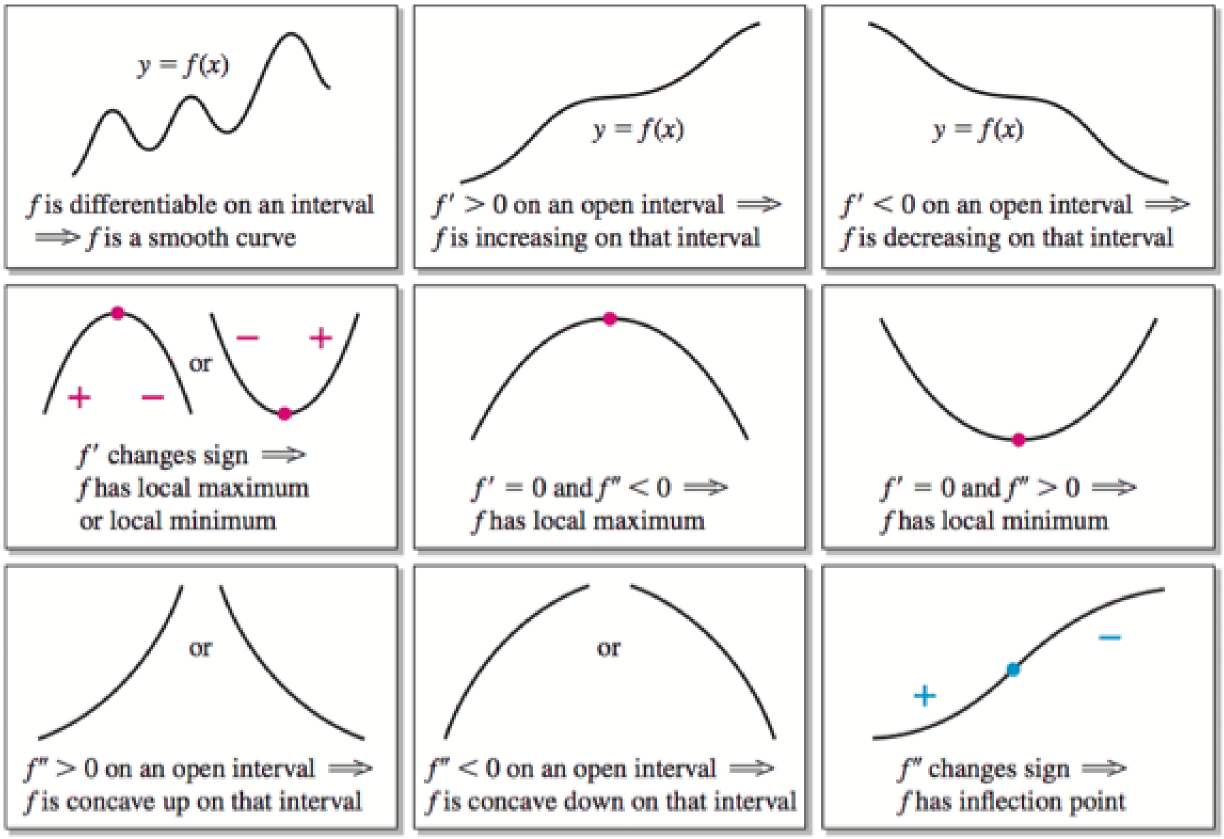
Solution. 1. The first derivative is $f'(x) = 12x - 3x^2 = 3x(4-x)$, so the critical points are at 0 and 4. The second derivative is $f''(x) = 12 - 6x$, so $f''(0) = 12$ and $f''(4) = -12$, so f has a local minimum at 0 and a local maximum at 4.

2. We have

$$p'(x) = \frac{e^x(x+1) - e^x}{(x+1)^2} = \frac{xe^x}{(x+1)^2},$$
$$p''(x) = \frac{(e^x + xe^x)(x+1)^2 - 2xe^x(x+1)}{(x+1)^4} = \frac{(e^x + xe^x)(x+1) - 2xe^x}{(x+1)^3} = \frac{e^x(x^2+1)}{(x+1)^3}.$$

So the only critical point of f is at 0. (Note that f is not defined at -1, so this is not a critical point.) Since $p''(0) = 1$, this is a local minimum.

Here is a recap of derivative properties:



2 Review

Let's review an assortment of techniques and problems from the course so far.

Example 3 (Related rates)**.** Two boats leave a dock at the same time. One boat travels south at 30 km/hr and the other travels east at 40 km/hr. After half an hour, how fast is the distance between the boats increasing?

Solution. Let x be the distance the first boat has traveled south from the dock, let y be the distance the second boat has traveled east from the dock, and let D be the distance between the boats. By the Pythagorean theorem,

$$x^2 + y^2 = D^2.$$

Differentiating, we obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2D\frac{dD}{dt}.$$

We are given $dx/dt = 30\,\text{km/hr}$ and $dy/dt = 40\,\text{km/hr}$. After half an hour, we have $x = 15\,\text{km}$ and $y = 20$ km; since $15^2 + 20^2 = 25^2$, we also have $D = 25$ km. So, after half an hour, we have

$$2(15\text{ km})(30\text{ km/hr}) + 2(20\text{ km})(40\text{ km/hr}) = 2(25\text{ km})\frac{dD}{dt}.$$

Solving for dD/dt yields

$$\frac{dD}{dt} = \frac{2 \cdot 15 \cdot 30 + 2 \cdot 20 \cdot 40}{2 \cdot 25}\,\mathrm{km/hr} = 50\,\mathrm{km/hr}.$$

Don't forget to check that this is a reasonable answer: the units are correct, the sign is correct, and the order of magnitude is plausible.

Example 4 (Inverse trig functions)**.** Find an equation for the line tangent to the graph of the function $f(x) = \cos^{-1}(x^2)$ at the point $(1/\sqrt{2}, \pi/3)$.

Solution. We compute the derivative:

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} \cdot 2x = \frac{-2x}{\sqrt{1-x^2}}.$$

Now we evaluate at the chosen point:

$$f'(1/\sqrt{2}) = \frac{-2/\sqrt{2}}{\sqrt{1-(1/\sqrt{2})^2}} = \frac{-\sqrt{2}}{\sqrt{1-1/2}} = \frac{-\sqrt{2}}{1/\sqrt{2}} = -2.$$

So the tangent line has slope -2 and passes through the point $(1/\sqrt{2}, \pi/3)$. An equation for this line if

$$y - \frac{\pi}{3} = -2\left(x - \frac{1}{\sqrt{2}}\right).$$

If you prefer slope-intercept form:

$$y = -2x + \frac{2}{\sqrt{2}} + \frac{\pi}{3} = -2x + \sqrt{2} + \frac{\pi}{3}.$$

Example 5 (Logarithmic and exponential functions)**.** Compute the derivative of the function $f(x) = \left(1 + \frac{1}{x}\right)^x$ on the domain $(0, \infty)$.

Solution. Note that $f(x) > 0$ for all $x > 0$. We express the function as an exponential with natural base:

$$f(x) = e^{\ln f(x)} = e^{x \ln(1+\frac{1}{x})}.$$

Now we take the derivative using the chain rule:

$$\begin{aligned} f'(x) &= e^{x\ln(1+\frac{1}{x})} \cdot \left(\ln(1+\tfrac{1}{x}) + x \cdot \frac{1}{1+\frac{1}{x}} \cdot (-x^{-2})\right) \\ &= e^{x\ln(1+\frac{1}{x})} \cdot \left(\ln(1+\tfrac{1}{x}) - \frac{1}{x+1}\right). \end{aligned}$$

Example 6 (Implicit differentiation)**.** Find an equation for the line tangent to the curve $\sin y + 5x = y^2$ at the point $(0, 0)$.

Solution. We use implicit differentiation and take the derivative with respect to x on both sides of the equation:

$$(\cos y)\frac{dy}{dx} + 5 = 2y\frac{dy}{dx}.$$

When $x = 0$ and $y = 0$, this becomes

$$(\cos 0)\frac{dy}{dx} + 5 = 0,$$

and since $\cos 0 = 1$, we obtain $dy/dx = -5$. So the tangent line has slope -5 and passes through the point $(0, 0)$; an equation for this line is $y = -5x$.

Example 7 (Chain rule)**.** Suppose $f(2) = 2$ and $f'(2) = 3$. Compute the derivative of $f(f(f(x)))$ at $x = 2$.

Solution. By the chain rule,

$$\frac{d}{dx}f(f(f(x))) = f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x).$$

Evaluating at $x = 2$, we obtain

$$f'(f(f(2))) \cdot f'(f(2)) \cdot f'(2) = f'(2) \cdot f'(2) \cdot f'(2) = 3 \cdot 3 \cdot 3 = 27.$$

Example 8 (Continuity)**.** Determine whether the following function is continuous from the left, from the right, both, or neither at $x = 1$:

$$f(x) = \begin{cases} 2x & \text{if } x < 1, \\ x^2 + 3x & \text{if } x > 1, \\ 4 & \text{if } x = 1. \end{cases}$$

Solution. We have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x = 2$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^2+3x) = 4$. Since $f(1) = 4$, this means that f is continuous from the right at 1, and not continuous from the left at 1. This also means that f is *not* continuous at 1.

Example 9 (Limits at infinity)**.** Compute the limit $\lim_{x\to\infty} \dfrac{40x^4 + x^2 + 5x}{\sqrt{64x^8 + x^6}}$.

Solution. We start by multiplying the numerator and denominator by x^{-4}:

$$\begin{aligned}
\lim_{x\to\infty} \frac{40x^4 + x^2 + 5x}{\sqrt{64x^8 + x^6}} &= \lim_{x\to\infty} \frac{x^{-4}(40x^4 + x^2 + 5x)}{x^{-4}\sqrt{64x^8 + x^6}} \\
&= \lim_{x\to\infty} \frac{40 + x^{-2} + 5x^{-3}}{\sqrt{x^{-8}}\sqrt{64x^8 + x^6}} \\
&= \lim_{x\to\infty} \frac{40 + x^{-2} + 5x^{-3}}{\sqrt{x^{-8}(64x^8 + x^6)}} \\
&= \lim_{x\to\infty} \frac{40 + x^{-2} + 5x^{-3}}{\sqrt{64 + x^{-2}}} \\
&= \frac{\lim_{x\to\infty}(40 + x^{-2} + 5x^{-3})}{\lim_{x\to\infty}(\sqrt{64 + x^{-2}})} \\
&= \frac{40}{\sqrt{64}} = \frac{40}{8} = 5.
\end{aligned}$$

Example 10 (Vertical asymptotes)**.** Find all vertical asymptotes of the function $f(x) = \dfrac{8x - x^2}{x^3 - 10x^2 + 16x}$. For each vertical asymptote a, determine $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$, and $\lim_{x\to a} f(x)$.

Solution. We factor the numerator and denominator:

$$f(x) = \frac{x(8-x)}{x(x^2 - 10x + 16)} = \frac{x(8-x)}{x(x-2)(x-8)} = \frac{1}{2-x}$$

for all real numbers x not equal to 0, 2, or 8. The only vertical asymptote is at $x = 2$; at 0 and 8, there is a removable singularity, not an asymptote. Indeed,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{2-x} = \frac{1}{2}, \qquad \lim_{x\to 8} f(x) = \lim_{x\to 8} \frac{1}{2-x} = -\frac{1}{6}.$$

As for the vertical asymptote at 2, note that $1/(2-x) > 0$ for $x < 2$ (and gets arbitrarily large as x approaches 2), and $1/(2-x) < 0$ for $x < 2$ (and gets arbitrarily large in magnitude as x approaches 2). So

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{2-x} = \infty, \qquad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{1}{2-x} = -\infty.$$

Since these two (infinite) limits do not have the same sign, $\lim_{x\to 2} f(x)$ does not exist and is not infinite.

Example 11 (Computing limits)**.** Let c be a positive real number. Compute

$$\lim_{x\to c} \frac{x^2 - c^2}{\sqrt{x} - \sqrt{c}}.$$

Solution. Using the difference of squares identity twice, we have

$$x^2 - c^2 = (x-c)(x+c) = (\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})(x+c).$$

So

$$\begin{aligned}
\lim_{x\to c} \frac{x^2 - c^2}{\sqrt{x} - \sqrt{c}} &= \lim_{x\to c} \frac{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})(x+c)}{\sqrt{x} - \sqrt{c}} \\
&= \lim_{x\to c} \left(\sqrt{x} + \sqrt{c}\right)(x+c) \\
&= \left(\sqrt{c} + \sqrt{c}\right)(c+c) = 4c\sqrt{c}.
\end{aligned}$$