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Today's topics

1 Optimization Problems 1

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Briggs–Cochran–Gillett §4.5, pp. 280–291

Now that we know what derivatives tell us, we can use this knowledge in *optimization* problems. The goal of optimization is finding the most *efficient* way to carry out a task, where efficient could mean least resource-intensive, most productive, least time-consuming, etc., depending on the situation. We will explore this in a set of examples, though the same ideas can be used in many situations!

Example 1 (§4.5, Ex. 7)**.** What two nonnegative real numbers with a sum of 23 have the largest possible product?

Solution. Let's approach the problem systematically.

- First, we identify and label the variables: Let a and b be the two numbers that we are summing.
- Next, we identify the *objective function* (that is, the function to be optimized): We want to maximize the product ab.
- Now we identify the constraint: $a + b = 23$.
- We can use the constraint to eliminate all but one independent variable of the objective function: We have $b = 23 - a$, so the objective function is equal to $a(23 - a)$.
- Next we identify the interval of interest for the objective function: Since both a and $b = 23 - a$ must be nonnegative, the domain is $[0, 23]$.
- Finally, we have a pure calculus problem: We want to find the absolute maximum of the function $P(a) = a(23 - a) = 23a - a^2$ on the interval $[0, 23]$. We know how to solve this sort of problem: The derivative is

  $$P'(a) = 23 - 2a,$$

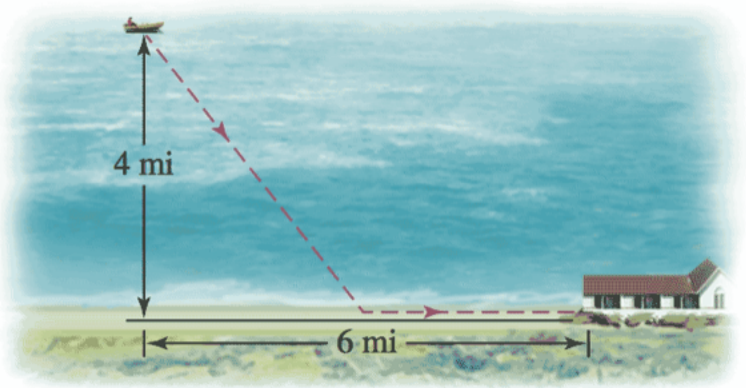
  which is zero exactly when $a = 23/2$. The second derivative is the constant function -2, so by the second derivative test, this is a local maximum. By Theorem 4.9 ("One Extremum Implies Absolute Extremum"), since this is the only local extremum of P on $[0, 23]$, this is in fact the absolute maximum. So $a = b = 23/2$ maximizes the product ab.

Example 2 (§4.5, Ex. 21: Shipping crates)**.** A square-based, box-shaped shipping crate is designed to have a volume of 16 ft^3. The material used to make the base costs twice as much (per square foot) as the material in the sides, and the material to do the top costs half as much (per square foot) as the material in the sides. What are the dimensions of the crate that minimize the cost of materials?

Solution. We use the same systematic approach:

- Identify the variables: Let s be the side length of the base of the crate (in feet), let h be the height of the crate (in feet), and let c be the cost per square foot of material in the sides.
- Identify the objective function: We want to minimize the cost of materials. The area of a side is sh, so the cost is csh. The area of the base is s^2, so the cost is $2cs^2$. The area of the top is also s^2, so the cost is $\frac{1}{2}cs^2$. Thus, the total cost is
$$csh + 2cs^2 + \tfrac{1}{2}cs^2 = c\left(sh + \tfrac{5}{2}s^2\right).$$
- Identify the constraint: The volume is 16 ft^3, so
$$s^2h = 16.$$
- Use the constraint to eliminate independent variables: We have $h = \frac{16}{s^2}$, so the objective function is
$$f(s) = c\left(s \cdot \frac{16}{s^2} + \frac{5}{2}s^2\right) = c\left(\frac{16}{s} + \frac{5s^2}{2}\right).$$
- Identify the interval of interest: s is a length and thus must be nonnegative, and also cannot be zero because then the constraint would have no solution. So the domain is $(0, \infty)$. Also note that c is a positive constant.
- Finally, we find the absolute maximum of f on $(0, \infty)$: We have
$$f'(s) = c\left(\frac{-16}{s^2} + 5s\right).$$
If $f'(s) = 0$, then (since $c > 0$), we obtain
$$5s = \frac{16}{s^2},$$
and thus $s^3 = 16/5$, so $s = (16/5)^{1/3}$. The second derivative is
$$f''(s) = c\left(\frac{32}{s^3} + 5\right),$$
which is positive, so f has a local minimum at $s = (16/5)^{1/3}$. Since this is the only local extremum, this is also the absolute minimum, and the optimal dimensions of the crate thus have side length $(16/5)^{1/3}$ and height $h = 16/s^2 = 16^{1/3} \cdot 5^{2/3}$.

Example 3 (§4.5, Ex. 27: Walking and rowing)**.** A boat on the ocean is 4 mi from the nearest point on a straight shoreline; that point is 6 mi from a restaurant on the shore. A woman plans to row the boat straight to some point on the shore and then walk to the restaurant.



1. If she walks at 3 mi/hr and rows at 2 mi/hr, at which point on the shore should she land to minimize the total travel time?

2. If she walks at 3 mi/hr, what is the minimum speed at which she must row so that the quickest way to the restaurant is to row directly (with no walking)?

Solution. 1. Let x be the distance (in miles) from the nearest point on the shoreline to the woman's landing point; we consider $0 \le x \le 6$ because the routes outside this interval involve backtracking and are clearly slower. So the woman rows for $\sqrt{x^2 + 4^2}$ miles (by the Pythagorean theorem), and walks for $6 - x$ miles. We want to minimize the travel time, which is

$$f(x) = \frac{\sqrt{x^2 + 16}}{2} + \frac{6 - x}{3}.$$

(The first term is the time in hours spent rowing, and the second term is the time in hours spent walking.) The derivative is

$$f'(x) = \frac{x}{2\sqrt{x^2 + 16}} - \frac{1}{3}.$$

If $f'(x) = 0$, then $\frac{1}{2}x = \frac{1}{3}\sqrt{x^2 + 16}$, so $9x^2 = 4(x^2 + 16)$, so $5x^2 = 64$, so $x = 8/\sqrt{5}$. The first derivative test implies that f has a local minimum at this point, and since this is the only critical value on $(0, 6)$, this gives the absolute minimum on $[0, 6]$.

2. Let $v > 0$ be the woman's rowing speed in miles per hour. Now the travel time is

$$f(x) = \frac{\sqrt{x^2 + 16}}{v} + \frac{6 - x}{3},$$

which has derivative

$$f'(x) = \frac{x}{v\sqrt{x^2 + 16}} - \frac{1}{3}.$$

Solving $f'(x) = 0$ as above, we see that f has at most one critical point on $(0, 6)$. Note that $f'(0) = -1/3$, so there cannot be an absolute minimum at 0. So there are two possibilities:

- There is a unique critical point of f in $(0, 6)$, which must give the absolute minimum of f on $[0, 6]$, and then $f'(6) > 0$; or
- f is decreasing on $[0, 6]$, and $f'(6) \leq 0$.

The latter case is when it's fastest to row directly to the restaurant. So we want to solve $f'(6) \leq 0$. We have

$$f'(6) = \frac{6}{v\sqrt{6^2 + 16}} - \frac{1}{3},$$

so the inequality is

$$v \geq \frac{6 \cdot 3}{\sqrt{6^2 + 16}} = \frac{18}{\sqrt{52}} = \frac{9}{\sqrt{13}} \text{ mi/hr.}$$