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Today's topics

1 Optimization Problems

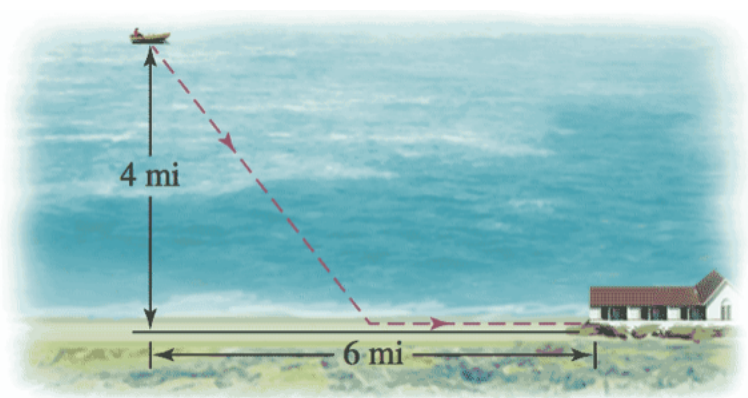
Briggs–Cochran–Gillett §4.5, pp. 280–291

Recall the general strategy for optimization problems we discussed last time:

1. Identify and label the variable quantities
2. Identify the objective function
3. Identify the constraints on the variables
4. Use the constraints to eliminate all but one of the variables in the objective function
5. Identify the domain that's relevant for the problem
6. Find the absolute maximum or minimum of the objective function on the domain

Let's see a few more optimization problems where this strategy is applicable.

Example 1 (§4.5, Ex. 27: Walking and rowing)**.** A boat on the ocean is 4 mi from the nearest point on a straight shoreline; that point is 6 mi from a restaurant on the shore. A woman plans to row the boat straight to some point on the shore and then walk to the restaurant.



If she walks at 3 mi/hr, what is the minimum speed at which she must row so that the quickest way to the restaurant is to row directly (with no walking)?

Solution. Let $v > 0$ be the woman's rowing speed in miles per hour. The travel time is

$$f(x) = \frac{\sqrt{x^2+16}}{v} + \frac{6-x}{3},$$

which has derivative

$$f'(x) = \frac{x}{v\sqrt{x^2+16}} - \frac{1}{3}.$$

Solving $f'(x) = 0$ as before, we obtain $3x = v\sqrt{x^2+16}$, so $9x^2 = v^2(x^2+16)$, so $(9-v^2)x^2 = 16v^2$. This has no solutions if $v \geq 3$, and has the unique solution $x = \frac{4v}{\sqrt{9-v^2}}$ if $v < 3$. Thus, f has at most one critical point on $(0,6)$. Note that $f'(0) = -1/3$, so there cannot be an absolute minimum at 0. So there are two possibilities:

- There is a unique critical point of f in $(0,6)$, which must give the absolute minimum of f on $[0,6]$, and then $f'(6) > 0$; or
- f is decreasing on $[0,6]$, and $f'(6) \leq 0$.

The latter case is when it's fastest to row directly to the restaurant. So we want to solve $f'(6) \leq 0$. We have

$$f'(6) = \frac{6}{v\sqrt{6^2+16}} - \frac{1}{3},$$

so the inequality is

$$v \geq \frac{6\cdot 3}{\sqrt{6^2+16}} = \frac{18}{\sqrt{52}} = \frac{9}{\sqrt{13}}\ \text{mi/hr}.$$

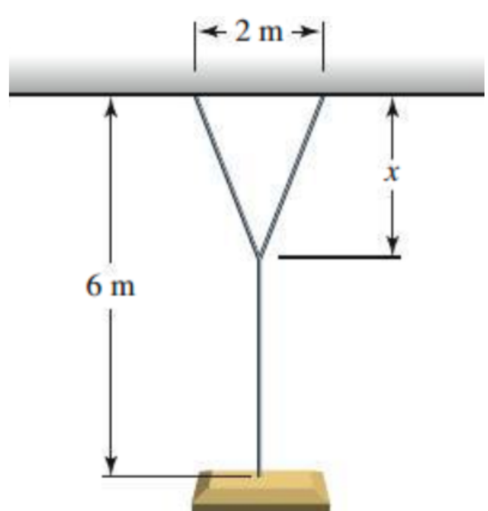
Example 4 (§4.5, Ex. 52: Suspension system)**.** A load must be suspended 6 m below a high ceiling using cables attached to two supports that are 2 m apart (see figure). How far below the ceiling (x in the figure) should the cables be joined to minimize the total length of cable used?

Solution. The two cables joined to the ceiling each have length $\sqrt{x^2+1}$ by the Pythagorean theorem, and the vertical segment of cable has length $6-x$. The objective function is the total length of cable:

$$L(x) = 2\sqrt{x^2+1} + 6 - x.$$

The lengths cannot be negative, so $0 \leq x \leq 6$. The derivative of the objective function is

$$L'(x) = \frac{2x}{\sqrt{x^2+1}} - 1,$$



which is defined everywhere on $[0, 6]$, so the critical points satisfy $L'(x) = 0$, which is equivalent to $2x = \sqrt{x^2+1}$. Then $4x^2 = x^2 + 1$, so $x = \sqrt{1/3} = \sqrt{3}/3$ is the unique critical point in $(0, 6)$. By the first (or second) derivative test, this is a local minimum, and by Theorem 4.9, this is therefore also the absolute minimum.

So the cables should be joined $\sqrt{3}/3 \approx 0.58\,\text{m}$ below the ceiling.

Example 2 (§4.5, Ex. 76: Turning a corner with a pole)**.** What is the length of the longest pole that can be carried horizontally around a corner at which a 3-ft corridor and a 4-ft corridor meet at right angles? What if there is an 8-ft ceiling and we allow the pole to be tilted at any angle?

Solution. The length of the longest pole that can be carried around the corner is the *shortest* length of a line segment that joins the outer walls while touching the inner corner. Using similar triangles, the length of such a segment is

$$L(x) = \sqrt{x^2+4^2} + \frac{3}{x}\sqrt{x^2+4^2} = \left(1 + \frac{3}{x}\right)\sqrt{x^2+16}.$$

We want to minimize this function for $x > 0$. The critical points of L satisfy

$$L'(x) = \frac{-3}{x^2}\sqrt{x^2+16} + \left(1+\frac{3}{x}\right)\frac{x}{\sqrt{x^2+16}} = 0,$$

which simplifies to $(x^2 + 3x)x = 3(x^2 + 16)$, so $x^3 = 48$. This gives a unique critical point $x = 48^{1/3}$ in $(0, \infty)$. By the first derivative test, this is a local minimum (for example, check $L'(1) < 0$ and $\lim_{x\to\infty} L'(x) > 0$). So by Theorem 4.9, this is also the absolute minimum. Thus, the length of the longest pole that can be carried horizontally around the corner is $48^{1/3} \approx 9.87\,\text{ft}$ long.

What if we allow the pole to be tilted at any angle? Observe that the *shadow* of the pole cast from a light source directly overhead has to satisfy the length requirement we just found above. So the maximum length of the pole is

$$\sqrt{8^2 + (48^{1/3})^2} = \sqrt{64 + 48^{2/3}}\ \text{ft}$$

by the Pythagorean theorem.