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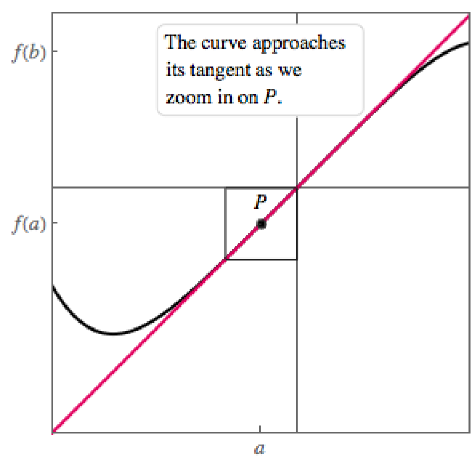
Today's topics

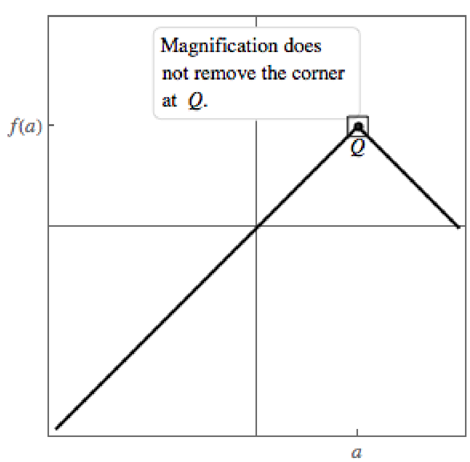
1 Linear approximation

Briggs–Cochran–Gillett §4.6, pp. 292–300

1.1 Tangent lines and the main idea

For a function that is differentiable at a point P, zooming in to the graph near P yields a piece of the curve that looks more and more like the tangent line at P. This fundamental observation, that smooth curves appear straighter on smaller scales, is called local linearity. It is the basis of many important mathematical ideas, one of which is *linear approximation*.





On the other hand, if we consider a curve with a corner or cusp at a point Q, no amount of magnification "straightens out" the curve at Q. The different behavior at P and Q is related to the idea of differentiability. One of the requirements for the techniques in this section is that the function be differentiable at the point in question.

The idea of linear approximation is to use the line tangent to the curve at P to approximate the value of the function at points near P.

Definition 1 (Linear approximation to f at a)**.** *Suppose f is differentiable on an interval I containing the point a. The linear approximation to f at a is the linear function*

$$L(x) = f(a) + f'(a)(x - a),$$

for x in I.

Example 2 (§4.6, Ex. 26)**.** Let $f(x) = \sin x$ and $a = \pi/4$.

(a) Write the equation of the line that represents the linear approximation to the function at the given point a.

(b) Use the linear approximation to estimate the value $f(0.75)$.

(c) Compute the percent error in your approximation,

$$100 \cdot \frac{|\text{approximation} - \text{exact}|}{|\text{exact}|},$$

where the exact value is given by a calculator.

Solution. The linear approximation to $\sin x$ at $\pi/4$ is

$$L(x) = \sin(\pi/4) + \cos(\pi/4)(x - \pi/4) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4).$$

This linear approximation gives the following value for $\sin(0.75)$:

$$L(0.75) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(0.75 - \pi/4) \approx 0.682076.$$

A more accurate approximation of $\sin(0.75)$ (given by a calculator) is 0.681639, giving a percent error of

$$100 \cdot \frac{0.682076 - 0.681639}{0.681639} \approx 0.064,$$

that is, an error of 0.064%, which is less than one part in 3000.

Example 3 (§4.6, Ex. 37, 40)**.** Use linear approximations to estimate the following quantities. Choose a value of a that produces a small error.

(a) $1/203$

(b) $\sqrt[3]{65}$

Solution. (a) We use the linear approximation of $f(x) = 1/x$ at 200. The derivative is $f'(x) = -1/x^2$, so the linear approximation is

$$L(x) = \frac{1}{200} - \frac{1}{200^2}(x - 200) = \frac{1}{100} - \frac{x}{200^2}.$$

The value of the linear approximation at 203 is

$$L(203) = \frac{1}{100} - \frac{203}{200^2} = 0.01 - 0.005075 = 0.004925.$$

Using a calculator, a more accurate approximation for $1/203$ is 0.004926, so the error in the linear approximation is only about 0.02%.

(b) We use the linear approximation of $f(x) = x^{1/3}$ at 64. The derivative is $f'(x) = \frac{1}{3}x^{-2/3}$, so the linear approximation is

$$L(x) = 64^{1/3} + \frac{1}{3 \cdot 64^{2/3}}(x - 64) = 4 + \frac{1}{48}(x - 64) = \frac{x}{48} + \frac{8}{3}.$$

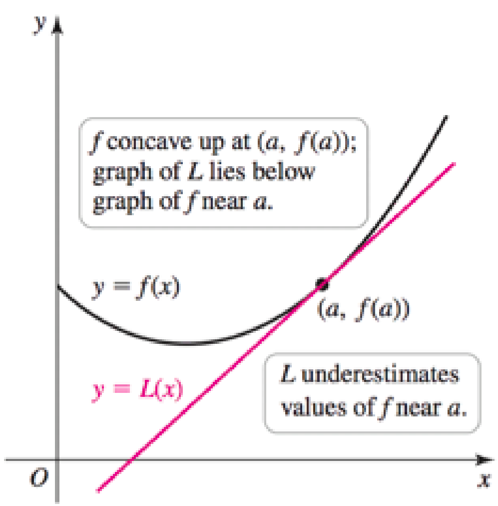
The value of the linear approximation at 65 is

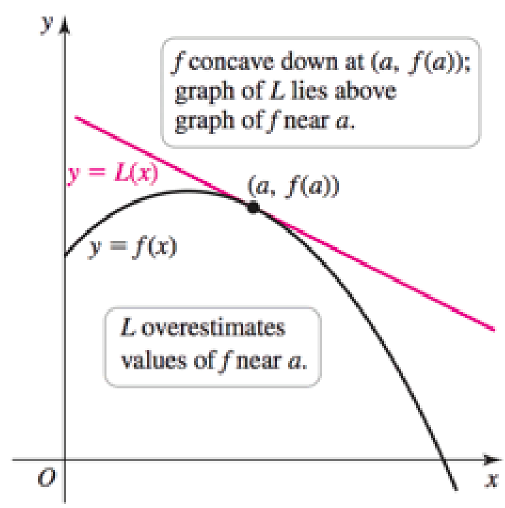
$$L(65) = \frac{65}{48} + \frac{8}{3} \approx 4.020833.$$

Using a calculator, a more accurate approximation for $\sqrt[3]{65}$ is 4.020725, so the error in the linear approximation is only about 0.003%.

1.2 Linear approximation and concavity

We can gain a more refined understanding of what a linear approximation is telling us by studying the function's concavity. Consider the following two graphs:





In the figure on the left, f is concave up on an interval containing a, and the graph of L lies below the graph of f near a. Consequently, L is an *underestimate* of the values of f near a. In the figure on the right, f is concave down on an interval containing a. Now the graph of L lies above the graph of f, which means that the linear approximation *overestimates* the values of f near a.

Example 4 (§4.6, Ex. 48)**.** Let $f(x) = 5 - x^2$ and $a = 2$.

(a) Find the linear approximation L to the function f at the point a.

(b) Graph f and L on the same set of axes.

(c) Based on the graph in part (1), state whether the linear approximation to f near a is an underestimate or overestimate.

(d) Compute $f''(a)$ to confirm your conclusion in part (3).

Solution. The linear approximation to $f(x) = 5 - x^2$ at 2 is

$$L(x) = f(2) + f'(2)(x-2) = 1 - 4(x-2) = 9 - 4x.$$

We have $f''(x) = -2$, so f is concave down. Thus, L is an overestimate, that is, $L(x) > f(x)$ for all x near (but not equal to) 2.

1.3 Change in y

We can also use the linear approximation of a function to study the change in y for a corresponding change in x.

Theorem 5 (Relationship between Δx and Δy)**.** *Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two points a and $a + \Delta x$ is approximately*

$$\Delta y \approx f'(a)\Delta x,$$

where $a + \Delta x$ is in I.

Example 6 (§4.6, Ex. 56)**.** Approximate the change in the atmospheric pressure when the altitude increases from $h = 2\,\text{km}$ to $h = 2.01\,\text{km}$. Use the following approximate formula for atmospheric pressure: $P(h) = 1000e^{-h/10}$ mbar.

Solution. The derivative is

$$P'(h) = -100e^{-h/10},$$

so $P'(2) = -100e^{-2/10} \approx -81.87\,\text{mbar/km}$. The linear approximation to the change in P near 2 is

$$\Delta P \approx P'(2)\Delta h.$$

For change from $h = 2\,\text{km}$ to $h = 2.01\,\text{km}$, we have $\Delta h = 0.01\,\text{km}$, so

$$\Delta P \approx P'(2) \cdot 0.01 \approx -0.8187\,\text{mbar}.$$

As a double-check, observe that the units are correct, the sign is correct (atmospheric pressure decreases as altitude increases), and the order of magnitude seems physically plausible. Using a calculator, a more accurate approximation for ΔP is -0.8183, for an error of about 0.05% in the linear approximation.