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Today's topics

Guest lecture by Prof. Glen Richard Hall today.

1 L'Hôpital's Rule

Briggs–Cochran–Gillett §4.7, pp. 301–312

In this section, we present a new technique, *l'Hôpital's Rule*, to evaluate certain limits called *indeterminate forms*.

What is an indeterminate form? As an example, consider $\lim_{x\to 0} \frac{\sin x}{x}$. If we attempt to substitute $x = 0$ into $\frac{\sin x}{x}$, we get $\frac{0}{0}$. Yet we earlier showed that $\frac{\sin x}{x}$ has limit 1 at $x = 0$. This limit is an example of an indeterminate form; in particular, $\lim_{x\to 0} \frac{\sin x}{x}$ has the indeterminate form $0/0$ because the numerator and denominator both approach 0 as $x \to 0$.

1.1 L'Hôpital's Rule for $0/0$

Theorem 1 (L'Hôpital's Rule)**.** *Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$). The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Example 2 (§4.7, Ex. 18, 22, 42, 48)**.** Evaluate the following limits using l'Hôpital's Rule:

1. $\displaystyle\lim_{x\to -1} \frac{x^4+x^3+2x+2}{x+1}$

Check: $(-1)^4+(-1)^3+2(-1)+2=0$ and $(-1)+1=0$. So

$$\lim_{x\to -1} \frac{x^4+x^3+2x+2}{x+1} = \lim_{x\to -1} \frac{4x^3+3x^2+2}{1} = 4(-1)^3+3(-1)^2+2 = 1.$$

2. $\displaystyle\lim_{x\to 0} \frac{e^x-1}{x^2+3x}$

Check: $e^0-1=0$ and $0^2+3\cdot 0=0$. Thus

$$\lim_{x\to 0} \frac{e^x-1}{x^2+3x} = \lim_{x\to 0} \frac{e^x}{2x+3} = \frac{e^0}{2\cdot 0+3} = \frac{1}{3}.$$

3. $\displaystyle\lim_{x\to\infty} \frac{\tan^{-1}x-\pi/2}{1/x}$

Check: $\lim_{x\to\infty}\tan^{-1}x-\pi/2=0$ and $\lim_{x\to\infty}1/x=0$. Hence

$$\lim_{x\to\infty} \frac{\tan^{-1}x-\pi/2}{1/x} = \lim_{x\to\infty} \frac{1/(1+x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{-x^2}{1+x^2} = -1.$$

4. $\displaystyle\lim_{y\to 2} \frac{y^2+y-6}{\sqrt{8-y^2}-y}$

Check: $2^2+2-6=0$ and $\sqrt{8-2^2}-2=0$. Therefore,

$$\lim_{y\to 2} \frac{y^2+y-6}{\sqrt{8-y^2}-y} = \lim_{y\to 2} \frac{2y+1}{-y(8-y^2)^{-1/2}-1} = \frac{2\cdot 2+1}{-2(8-2^2)^{-1/2}-1} = \frac{5}{-2}.$$

1.2 L'Hôpital's Rule for ∞/∞

L'Hôpital's Rule also applies directly to limits of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$, where $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. This indeterminate form is denoted ∞/∞.

Theorem 3 (L'Hôpital's Rule for ∞/∞)**.** *Suppose that f and g are differentiable on an open interval I containing a, with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists (or is $\pm\infty$). The rule also applies for $x \to \pm\infty, x \to a^+$, or $x \to a^-$.

Example 4 (§4.7, Ex. 24, 46)**.** Evaluate the following limits:

1. $\lim_{x\to\infty}\frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$

$$\lim_{x\to\infty}\frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4} = \lim_{x\to\infty}\frac{12x^2 - 4x}{3\pi x^2} = \lim_{x\to\infty}\frac{24x - 4}{6\pi x} = \lim_{x\to\infty}\frac{24}{6\pi} = \frac{4}{\pi}$$

2. $\lim_{x\to\infty}\frac{\ln(3x + 5e^x)}{\ln(7x + 3e^{2x})}$

Check: $\lim_{x\to\infty}\ln(3x + 5e^x) = \infty$ and $\lim_{x\to\infty}\ln(7x + 3e^{2x}) = \infty$. So

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln(3x + 5e^x)}{\ln(7x + 3e^{2x})} &= \lim_{x\to\infty}\frac{(3 + 5e^x)/(3x + 5e^x)}{(7 + 6e^{2x})/(7x + 3e^{2x})}\\ &= \lim_{x\to\infty}\frac{15e^{3x} + 9e^{2x} + 35xe^x + 21x}{30e^{3x} + 18xe^{2x} + 35e^x + 21x}\cdot\frac{e^{-3x}}{e^{-3x}}\\ &= \lim_{x\to\infty}\frac{15 + 9e^{-x} + 35xe^{-2x} + 21xe^{-3x}}{30 + 18xe^{-x} + 35e^{-2x} + 21xe^{-3x}} = \frac{15}{30} = \frac{1}{2}.\end{aligned}$$

1.3 How to handle related indeterminate forms: $0\cdot\infty$ and $\infty - \infty$

Here we consider two other indeterminate forms. Limits of the form $\lim_{x\to a} f(x)g(x)$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$ are denoted $0\cdot\infty$, while limits of the form $\lim_{x\to a}(f(x) - g(x))$, where $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ are indeterminate forms we denote $\infty - \infty$. ***L'Hôpital's Rule cannot directly be applied to either of these limits! However, if we can recast these indeterminate forms into the form $0/0$ or ∞/∞, then we may apply L'Hôpital's Rule.***

Occasionally, it helps to convert a limit as $x \to \infty$ to a limit as $t \to 0^+$ (or vice versa) by a change of variables. To evaluate $\lim_{x\to\infty} f(x)$, we define $t = \frac{1}{x}$ and note that as $x \to \infty$, we have $t \to 0^+$. Then $\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f\left(\frac{1}{t}\right)$.

Example 5 (§4.7, Ex. 54, 57, 61, 64)**.** Evaluate the following limits:

1. $\lim_{x\to 1^-} (1-x)\tan\frac{\pi x}{2}$

$$\lim_{x\to 1^-} (1-x)\tan\frac{\pi x}{2} = \lim_{x\to 1^-} \frac{(1-x)\sin(\frac{\pi}{2}x)}{\cos(\frac{\pi}{2}x)}$$
$$= \lim_{x\to 1^-} \frac{-\sin(\frac{\pi}{2}x) + (1-x)\frac{\pi}{2}\cos(\frac{\pi}{2}x)}{-\frac{\pi}{2}\sin(\frac{\pi}{2}x)} = \frac{-\sin(\frac{\pi}{2})}{-\frac{\pi}{2}\sin(\frac{\pi}{2})} = \frac{2}{\pi}$$

2. $\lim_{x\to \pi/2^-} \left(\frac{\pi}{2} - x\right)\sec x$

$$\lim_{x\to \pi/2^-} \left(\frac{\pi}{2} - x\right)\sec x = \lim_{x\to \pi/2^-} \frac{\frac{\pi}{2} - x}{\cos x} = \lim_{x\to \pi/2^-} \frac{-1}{-\sin x} = \frac{-1}{-\sin\frac{\pi}{2}} = 1$$

3. $\lim_{x\to 0^+} \left(\cot x - \frac{1}{x}\right)$

We apply l'Hôpital's rule twice:

$$\lim_{x\to 0^+} \left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0^+} \frac{x\cos x - \sin x}{x\sin x} = \lim_{x\to 0^+} \frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}$$
$$= \lim_{x\to 0^+} \frac{-x\sin x}{\sin x + x\cos x} = \lim_{x\to 0^+} \frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x}$$
$$= \frac{-\sin 0 - 0\cos 0}{2\cos 0 - 0\sin 0} = \frac{-0-0}{2-0} = 0.$$

4. $\lim_{x\to\infty} \left(x - \sqrt{x^2+4x}\right)$

$$\lim_{x\to\infty} \left(x - \sqrt{x^2+4x}\right) = \lim_{x\to 0^+} \left(\frac{1}{x} - \sqrt{\frac{1}{x^2} + \frac{4}{x}}\right) = \lim_{x\to 0^+} \frac{1-\sqrt{1+4x}}{x}$$
$$= \lim_{x\to 0^+} \frac{-2(1+4x)^{-1/2}}{1} = -2(1+0)^{-1/2} = -2.$$