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Today's topics

1 L'Hôpital's Rule

Briggs–Cochran–Gillett §4.7, pp. 301–312

1.1 L'Hôpital's Rule for ∞/∞

Let's see a few more examples of l'Hôpital's rule applied to limits at infinity.

Example 1 (§4.7, Ex. 24, 46)**.** Evaluate the following limits:

1. $\lim\limits_{x\to\infty} \dfrac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$

We could use the theorem on limits at infinity of rational functions, but let's see how we can solve this by repeatedly applying l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4} = \lim_{x\to\infty} \frac{12x^2 - 4x}{3\pi x^2} = \lim_{x\to\infty} \frac{24x - 4}{6\pi x} = \lim_{x\to\infty} \frac{24}{6\pi} = \frac{4}{\pi}$$

2. $\lim\limits_{x\to\infty} \dfrac{\ln(3x + 5e^x)}{\ln(7x + 3e^{2x})}$

Check: $\lim_{x\to\infty} \ln(3x + 5e^x) = \infty$ and $\lim_{x\to\infty} \ln(7x + 3e^{2x}) = \infty$. So

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(3x + 5e^x)}{\ln(7x + 3e^{2x})} &= \lim_{x\to\infty} \frac{(3 + 5e^x)/(3x + 5e^x)}{(7 + 6e^{2x})/(7x + 3e^{2x})} \\ &= \lim_{x\to\infty} \frac{15e^{3x} + 9e^{2x} + 35xe^x + 21x}{30e^{3x} + 18xe^{2x} + 35e^x + 21x} \cdot \frac{e^{-3x}}{e^{-3x}} \\ &= \lim_{x\to\infty} \frac{15 + 9e^{-x} + 35xe^{-2x} + 21xe^{-3x}}{30 + 18xe^{-x} + 35e^{-2x} + 21xe^{-3x}} = \frac{15}{30} = \frac{1}{2}.\end{aligned}$$

3. $\lim\limits_{x\to\infty} \dfrac{e^{1/x}}{e^{3/x}}$

We have $\lim_{x\to\infty} e^{1/x} = e^0 = 1$, so we *cannot* use l'Hôpital's rule here! In fact, the limit is just 1, since $\lim_{x\to\infty} e^{3/x} = e^0 = 1$ as well. But if we incorrectly apply l'Hôpital's rule, we'd get the wrong answer:

$$\lim_{x\to\infty} \frac{e^{1/x}}{e^{3/x}} \neq \lim_{x\to\infty} \frac{-x^{-2}e^{1/x}}{-3x^{-2}e^{3/x}} = \frac{1}{3}\lim_{x\to\infty} \frac{e^{1/x}}{e^{3/x}} = \frac{1}{3}.$$

1.2 How to handle related indeterminate forms: $0 \cdot \infty$ and $\infty - \infty$

Here we consider two other indeterminate forms. Limits of the form $\lim\limits_{x\to a} f(x)g(x)$, where $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \pm\infty$ are denoted $0 \cdot \infty$, while limits of the form $\lim\limits_{x\to a}(f(x) - g(x))$, where $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$ are indeterminate forms we denote $\infty - \infty$. ***L'Hôpital's Rule cannot directly be applied to either of these limits! However, if we can recast these indeterminate forms into the form*** $0/0$ ***or*** ∞/∞***, then we may apply L'Hôpital's Rule.***

Occasionally, it helps to convert a limit as $x \to \infty$ to a limit as $t \to 0^+$ (or vice versa) by a change of variables. To evaluate $\lim\limits_{x\to\infty} f(x)$, we define $t = \frac{1}{x}$ and note that as $x \to \infty$, we have $t \to 0^+$. Then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{t\to 0^+} f\left(\dfrac{1}{t}\right)$.

Example 2 (§4.7, Ex. 54, 57, 61, 64)**.** Evaluate the following limits:

1. $\lim\limits_{x\to 1^-} (1 - x)\tan\dfrac{\pi x}{2}$

$$\begin{aligned}\lim_{x\to 1^-} (1-x)\tan\frac{\pi x}{2} &= \lim_{x\to 1^-} \frac{(1-x)\sin(\frac{\pi}{2}x)}{\cos(\frac{\pi}{2}x)} \\ &= \lim_{x\to 1^-} \frac{-\sin(\frac{\pi}{2}x) + (1-x)\frac{\pi}{2}\cos(\frac{\pi}{2}x)}{-\frac{\pi}{2}\sin(\frac{\pi}{2}x)} = \frac{-\sin(\frac{\pi}{2})}{-\frac{\pi}{2}\sin(\frac{\pi}{2})} = \frac{2}{\pi}\end{aligned}$$

2. $\lim\limits_{x\to \pi/2^-} \left(\dfrac{\pi}{2} - x\right)\sec x$

$$\lim_{x\to\pi/2^-} \left(\frac{\pi}{2} - x\right)\sec x = \lim_{x\to\pi/2^-} \frac{\frac{\pi}{2} - x}{\cos x} = \lim_{x\to\pi/2^-} \frac{-1}{-\sin x} = \frac{-1}{-\sin\frac{\pi}{2}} = 1$$

3. $\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right)$

We apply l'Hôpital's rule twice:

$$\begin{aligned}\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) &= \lim_{x\to 0^+}\frac{x\cos x - \sin x}{x\sin x} = \lim_{x\to 0^+}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x}\\ &= \lim_{x\to 0^+}\frac{-x\sin x}{\sin x + x\cos x} = \lim_{x\to 0^+}\frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x}\\ &= \frac{-\sin 0 - 0\cos 0}{2\cos 0 - 0\sin 0} = \frac{-0-0}{2-0} = 0.\end{aligned}$$

4. $\lim_{x\to\infty}\left(x - \sqrt{x^2+4x}\right)$

$$\begin{aligned}\lim_{x\to\infty}\left(x - \sqrt{x^2+4x}\right) &= \lim_{x\to 0^+}\left(\frac{1}{x} - \sqrt{\frac{1}{x^2}+\frac{4}{x}}\right) = \lim_{x\to 0^+}\frac{1-\sqrt{1+4x}}{x}\\ &= \lim_{x\to 0^+}\frac{-\frac{1}{2}(1+4x)^{-1/2}}{1} = -\frac{1}{2}(1+0)^{-1/2} = -\frac{1}{2}\end{aligned}$$

1.3 Indeterminate forms 1^∞, 0^0 and ∞^0

The indeterminate forms $1^\infty,\ 0^0,\ \infty^0$ arise from limits of the form

$$\lim_{x\to a,\pm\infty,a^+,a^-} f(x)^{g(x)}.$$

L'Hôpital's rule cannot be applied directly, but we can rewrite the expression inside the limit as an exponential:

$$f(x)^{g(x)} = e^{g(x)\ln(f(x))}.$$

Using this and the fact that the exponential function is continuous we know that

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln(f(x))} = e^{\lim_{x\to a} g(x)\ln(f(x))}.$$

So we just need to worry about the limit $\lim_{x\to a} g(x)\ln(f(x))$. Let us illustrate this in some examples.

Example 3 (§4.7, Ex. 76, 81, 82)**.** Evaluate the following limits:

1. $\lim_{x\to 0}(1+4x)^{3/x}$

By l'Hôpital's rule,

$$\lim_{x\to 0}\frac{3\ln(1+4x)}{x}=\lim_{x\to 0}\frac{12/(1+4x)}{1}=\frac{12}{1+4\cdot 0}=12.$$

So

$$\lim_{x\to 0}(1+4x)^{3/x}=\lim_{x\to 0}e^{\frac{3}{x}\ln(1+4x)}=e^{12}.$$

2. $\lim_{x\to 0}(e^{5x}+x)^{1/x}$

$$\lim_{x\to 0}\frac{\ln(e^{5x}+x)}{x}=\lim_{x\to 0}\frac{5e^{5x}+1}{e^{5x}+x}=\frac{5+1}{1+0}=6,$$

so

$$\lim_{x\to 0}(e^{5x}+x)^{1/x}=\lim_{x\to 0}e^{\frac{1}{x}\ln(e^{5x}+x)}=e^{6}.$$

3. $\lim_{z\to\infty}\left(1+\frac{10}{z^2}\right)^{z^2}$

$$\lim_{z\to\infty}z^2\ln\left(1+\frac{10}{z^2}\right)=\lim_{z\to\infty}\frac{\ln(1+\frac{10}{z^2})}{z^{-2}}=\lim_{z\to\infty}\frac{-20z^{-3}/(1+\frac{10}{z^2})}{-2z^{-3}}=\lim_{z\to\infty}\frac{10}{1+\frac{10}{z^2}}=10,$$

so

$$\lim_{z\to\infty}\left(1+\frac{10}{z^2}\right)^{z^2}=\lim_{x\to 0}e^{z^2\ln(1+10/z^2)}=e^{10}.$$