Dr. Daniel Hast, *drhast@bu.edu*

Today's topics

1 L'Hôpital's rule

1.1 Summary of indeterminate forms

Here are all the indeterminate forms we've discussed:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 1^\infty,\ 0^0,\ \infty^0.$$

These fall into three classes:

- $0/0$ and ∞/∞: We can *directly* apply l'Hôpital's rule to limits of this form.
- $0\cdot\infty$ and $\infty-\infty$: We cannot directly apply l'Hôpital's rule, but often we can use algebra to rewrite the limit as a fraction of the form $0/0$ or ∞/∞.
- 1^∞, 0^0, and ∞^0: We cannot directly apply l'Hôpital's rule, but we can use the identity $f(x)^{g(x)} = e^{g(x)\ln f(x)}$ and then try to compute the limit of $g(x)\ln f(x)$.

Limits that are **NOT** indeterminate forms:

- $1/\infty$ (the limit is 0)
- $1/0$ (the limit doesn't exist, and may be $\pm\infty$)
- $0/\infty$ (the limit is 0)
- $\infty/0$ (the limit doesn't exist, and may be $\pm\infty$)
- 0^∞ (the limit is 0)

1.2 Growth rates

Our goal now is to use what we know about limits, including l'Hôpital's rule, to obtain *a ranking* of the functions we know **based on their growth rates**.

Definition 1 (Growth Rates of functions as $x \to \infty$)**.** *Let f and g be functions such that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Then*

- *f **grows faster than** g as $x \to \infty$ if*

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \text{ or, equivalently, } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

  We write $g(x) \ll f(x)$.

- *f and g **have comparable growth rates** if*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M,$$

  with $M > 0$.

We have:

$$\ln x \ll x^p \ll e^x \qquad \text{for all } p > 0.$$

More generally,

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll e^x,$$

where $q, p, r > 0$.

Example 2. Use limits to determine which of the following functions grows faster or state they have comparable growth rates:

(a) $x^2 \ln x$ and $\ln^5 x$

(b) 100^x and x^x

(c) $\ln x$ and $\ln(\ln x)$

(d) e^{x^2} and x^x

2 Antiderivatives

Briggs–Cochran–Gillett §4.9, pp. 321–327

2.1 Definition and first examples

The reverse process to differentiation is called **antidifferentiation**.

Definition 3. *A function F is an* ***antiderivative*** *of f if $F'(x) = f(x)$.*

For example, $F(x) = x+5$ is an antiderivative of $f(x) = 1$ because $F'(x) = f(x)$. Likewise,

$$\frac{d}{dx}e^x = e^x,$$

so $F(x) = e^x$ is an antiderivative of $f(x) = e^x$. Of course, we also have

$$\frac{d}{dx}(e^x - 2) = e^x,$$

so $F_1(x) = e^x - 2$ is another antiderivative of $f(x) = e^x$. As we saw a few weeks ago, a consequence of the MVT is that

a function $f(x)$ differs from another function $g(x)$ by a constant
if and only if $f'(x) = g'(x)$.

This proves the following theorem:

Theorem 4. *Let F be an antiderivative of f on an interval I. Then all the antiderivatives of f on I have the form $F(x) + C$ where C is an arbitrary constant.*

Remark 5. If the domain of f is not an interval, then there can be more antiderivatives—we can take a different constant on different contiguous pieces of the domain. For example, since $f(x) = 1/x$ is defined on $(-\infty, 0)$ and $(0, \infty)$, but not at 0, one antiderivative of f is $\ln|x|$, but another antiderivative is

$$F(x) = \begin{cases} \ln|x| + 7 & \text{if } x > 0, \\ \ln|x| - 11 & \text{if } x < 0. \end{cases}$$

Definition 6. *Let F be an antiderivative of a function f on an interval I. We write*

$$\int f(x)\,dx = F(x) + C$$

and say that $F(x) + C$ is the ***indefinite integral*** *of f.*

Example 7 (§4.9, Ex. 12, 14, (based on) 19, 20)**.** Find all antiderivatives of the following functions. Check your work by taking derivatives.

1. $g(x) = 11x^{10}$
2. $g(x) = -4\cos(4x)$
3. $f(x) = 2e^{2x}$
4. $h(y) = y^{-1}$

2.2 Indefinite integrals to know

It is useful to know some of the more common indefinite integrals:

- $\int K dx = Kx + C$
- $\int x^p dx = \frac{x^{p+1}}{p+1} + C$, if $p \neq -1$
- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int \cos(ax) dx = \frac{1}{a}\sin(ax) + C$
- $\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + C$
- $\int \sec^2(ax) dx = \frac{1}{a}\tan(ax) + C$
- $\int \sec(ax)\tan(ax) dx = \frac{1}{a}\sec(ax) + C$
- $\int e^{ax} dx = \frac{1}{a}e^{ax} + C$
- $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\left|\frac{x}{a}\right|\right) + C$

Also, by the corresponding rules for differentiation, we have that

- $\int cf(x) dx = c \int f(x) dx$
- $\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx$