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Today's topics

1 Antiderivatives

Briggs–Cochran–Gillett §4.9, pp. 321–327

Example 1. Determine the following indefinite integrals. Check your work by taking derivatives.

(a) $\int (3u^{-2} - 4u^2 + 1)du = -3u^{-1} - \frac{4}{3}u^3 + u + C$

(b) $\int (3x + 1)(4 - x)dx = \int (11x - 3x^2 + 4)dx = \frac{11}{2}x^2 - x^3 + 4x + C$

(c) $\int (\sin(4t) - \sin(t/4))\, dt = -\frac{1}{4}\cos(4t) + 4\cos(t/4) + C$

(d) $\int \frac{3}{4 + v^2}dv$

(e) $\int e^{x+2}dx$

(f) $\int \frac{10t^5 - 3}{t}dt$

(g) $\int 2\sec^2(2v)dv$

For (d), if we recall that $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$, then we can see that

$$\int \frac{3}{4 + v^2}dv = a\tan^{-1}(bv) + C$$

for some constants a and b. Let's determine these constants by taking the derivative:

$$\frac{d}{dv}\left(a\tan^{-1}(bv) + C\right) = a \cdot \frac{1}{1 + (bv)^2} \cdot b = \frac{ab}{1 + (bv)^2} = \frac{4ab}{4 + (2bv)^2} = \frac{3}{4 + v^2}.$$

To get the denominators to match, we need $b = \frac{1}{2}$. Then to get the numerators to match, we need $4ab = 3$, so $a = \frac{3}{2}$. Thus

$$\int \frac{3}{4+v^2} dv = \frac{3}{2} \tan^{-1} \frac{v}{2} + C$$

(Alternatively, if you know the antiderivative formula $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$, then you can get the above in one step. But it's still a good idea to know how to do it the longer way, in case you don't remember the more general formula.)

2 Introduction to differential equations

Briggs–Cochran–Gillett §4.9, pp. 327–334

2.1 Initial value problems

An equation involving an unknown function and its derivatives is called a *differential equation*. For example, suppose you know that the derivative of a function f satisfies the equation

$$f'(x) = 2x + 10.$$

To find a function f that satisfies this equation, we note that the solutions are antiderivatives of $2x + 10$, which are

$$f(x) = x^2 + 10x + C,$$

where C is an arbitrary constant.

Now suppose we further gave an *initial condition* of

$$f(1) = 13;$$

this then would allow us to determine the constant. By substituting in 1, we find

$$f(1) = 1^2 + 10 \cdot 1 + C = 13$$

which implies that $C = 2$. This then gives the solution that

$$f(x) = x^2 + 10x + 2.$$

A differential equation together with an initial condition is called an *initial value problem.*

Example 2. For the following functions f, find the antiderivative F that satisfies the given condition.

1. $f(x) = \frac{4\sqrt{x}+6/\sqrt{x}}{x^2}$; $F(1) = 4$.
2. $f(\theta) = 2\sin(2\theta) - 4\cos(4\theta)$; $F\left(\frac{\pi}{4}\right) = 2$.

Solution.

1. We have $f(x) = 4x^{-3/2} + 6x^{-5/2}$, so
$$F(x) = -8x^{-1/2} - 4x^{-3/2} + C,$$
which you can confirm by taking the derivative. The initial value gives us
$$4 = F(1) = -8 - 4 + C,$$
so $C = 16$ and
$$F(x) = -8x^{-1/2} - 4x^{-3/2} + 16.$$

2. We have $F(x) = -\cos(2\theta) - \sin(4\theta) + C$, and the initial value gives
$$2 = F(\pi/4) = -\cos(\pi/2) - \sin(\pi) + C = -0 - 0 + C,$$
so $C = 2$ and
$$F(x) = -\cos(2\theta) - \sin(4\theta) + 2.$$

Example 3. Find the solution of the following initial value problems.

1. $y'(t) = \frac{3}{t} + 6; y(1) = 8$
2. $u'(x) = \frac{e^{2x}+4e^{-x}}{e^x}; u(\ln 2) = 2$

2.2 One-dimensional motion

Antiderivatives allow us to revisit the topic of one-dimensional motion. Suppose the position of an object that moves along a line relative to an origin is $s(t)$, where $t \geq 0$ measures elapsed time. The velocity of the object is $v(t) = s'(t)$, which we now re-interpret in terms of antiderivatives: *The position function is an antiderivative of the velocity.* If we are given the velocity function of an object and its position at a particular time, we can determine its position at all future times by solving an initial value problem.

Moreover, we know that the acceleration $a(t)$ of an object moving in one dimension satisfies $a(t) = v'(t)$. This says that velocity is an antiderivative of the acceleration. So if we are given the acceleration of an object and its velocity at a particular time, we can determine its velocity at all times. To summarize:

Theorem 4 (Initial value problems for velocity and position)**.** *Suppose an object moves along a line with a velocity $v(t)$ for $t \geq 0$. Then its position is found by solving the initial value problem*
$$s'(t) = v(t), s(0) = s_0, \quad \textit{where } s_0 \textit{ is the initial position.}$$
If the acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem
$$v'(t) = a(t), v(0) = v_0, \quad \textit{where } v_0 \textit{ is the initial velocity.}$$

Example 5. Given the velocity function $v(t) = e^{-2t} + 4$ of an object moving along a line, find the position function with the given initial position $s(0) = 2$. Then graph both the velocity and position function.

Solution. The position function is an antiderivative of velocity:

$$s(t) = -\frac{1}{2}e^{-2t} + 4t + C.$$

The initial value yields $2 = s(0) = -\frac{1}{2}e^0 + 0 + C = -\frac{1}{2} + C$, so $C = \frac{5}{2}$.

Example 6. Given the acceleration function $a(t) = 4$ of an object moving along a line, find the position function with the following given initial velocity and position: $v(0) = -3, s(0) = 2$.

Example 7. Consider the following description of the vertical motion of an object subject only to the acceleration due to gravity: A stone is thrown vertically upward with a velocity of 30 m/s from the edge of a cliff 200 m above a river.

Begin with the acceleration equation $a(t) = v'(t) = g$, where $g = -9.8m/s^2$.

(a) Find the velocity of the object for all relevant times.

(b) Find the position of the object for all relevant times.

(c) Find the time when the object reaches its highest point. What is the height?

(d) Find the time when the object strikes the ground.

Solution. (a) We have $v(t) = gt + C$ and $v(0) = 30$, so $v(t) = gt + 30$.

(b) We have $s(t) = \frac{1}{2}gt^2 + 30t + D$ (where D is a constant), and $s(0) = 200$, so

$$s(t) = \frac{1}{2}gt^2 + 30t + 200.$$

(c) The object reaches its highest point when $v(t) = 0$, that is, at $t = \frac{30}{-g} = \frac{30}{9.8} \approx 3.0612$ seconds.

(d) The object strikes the ground when $s(t) = 0$ and $t > 0$. We have

$$s(t) = -4.9t^2 + 30t + 200,$$

which by the quadratic formula has roots

$$\frac{-30 \pm \sqrt{30^2 + 4 \cdot 4.9 \cdot 200}}{-9.8} = \frac{30 \pm \sqrt{4820}}{9.8}.$$

These roots are approximately equal to -4.0231 and 10.1455, so the object strikes the ground after about 10.1455 seconds.