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Today's topics

1 Approximating area under curves and Riemann sums

Briggs–Cochran–Gillett §5.1, pp. 338–352

1.1 Introduction

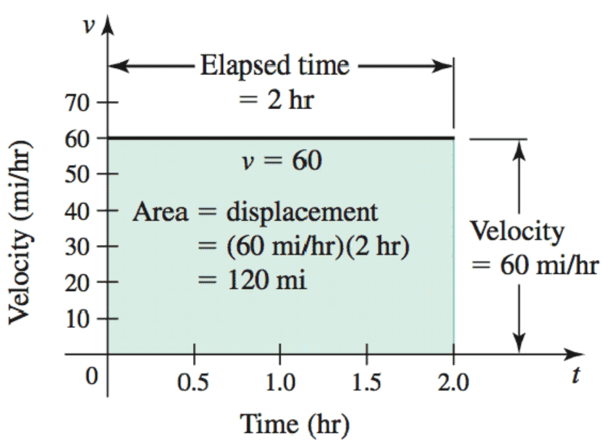
Geometrically, we know that if we graph the displacement function $s(t)$, the derivative function—which gives the velocity $v(t)$ at each instant—gives the slope of the tangent to each point of the graph of $s(t)$.

We also know that if we start with the velocity function $v(t)$, we can compute its antiderivative to find $s(t)$. What can we say about this geometrically? If we have the graph of $v(t)$, how can we find $s(t)$? This is an important question that we are going to explore during the next few classes.

If the velocity is constant, the answer is simple, and it points us in the right direction: for example, consider a car moving in a line at constant velocity of 60 mi/hr. Then the displacement is simply the velocity times the time. If the car drove for two hours, then

$$s(2) = 60 \times 2 = 120 \text{ mi}.$$

Geometrically, this is the area under the graph of the velocity:

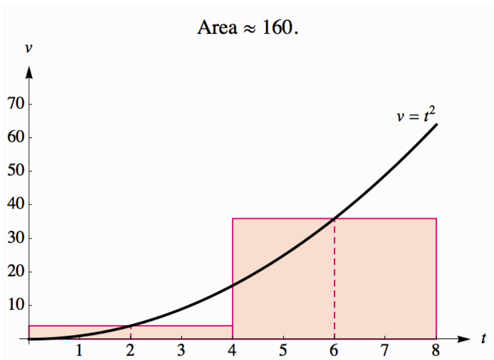


If the velocity is not constant, the displacement is not simply velocity times time. But what if we subdivided the time into intervals where we could approximate the velocity as being constant? Then we would be able to calculate and approximate displacement!

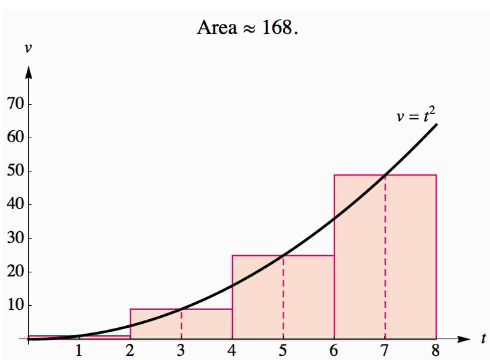
1.2 Approximating area under curves

Consider a car moving with velocity $v(t) = t^2$ mi/hr, from $t = 0$ to $t = 8$.

- Subdivision into 2 intervals:



- Subdivision into 4 intervals:



To get better approximations, we could continue to subdivide the interval: see interactive Figure 5.5, Chapter 5.1 of the textbook.

In this case we used the **midpoint** value for an approximate value of the velocity on the interval. We could also have used the **left endpoint** or the **right endpoint** (or any other point really...)

Example 1 (§5.1, Ex. 17)**.** If the velocity of an object is $v(t) = 2t + 1$ (m/s), approximate the displacement of the object on $0 \le t \le 8$ by subdividing the interval in 2 subintervals. **Use the left endpoint of each subinterval to compute the height of the rectangles**.

Solution. We subdivide into intervals $[0, 4]$ and $[4, 8]$. The values at the left endpoints are $v(0) = 1$ and $v(4) = 9$, and the subintervals each have width $\Delta t = 4$, so the left Riemann sum is

$$v(0) \cdot \Delta t + v(4) \cdot \Delta t = 1 \cdot 4 + 9 \cdot 4 = 4 + 36 = 40.$$

1.3 Riemann sums

The process of **approximating the area below a curve** using areas of rectangles is known as computing **Riemann sums**. This is useful for much more than computing displacements and deserves our attention.

Definition 2 (Regular Partition). *Given an interval $[a, b]$ and a positive integer n, we compute*

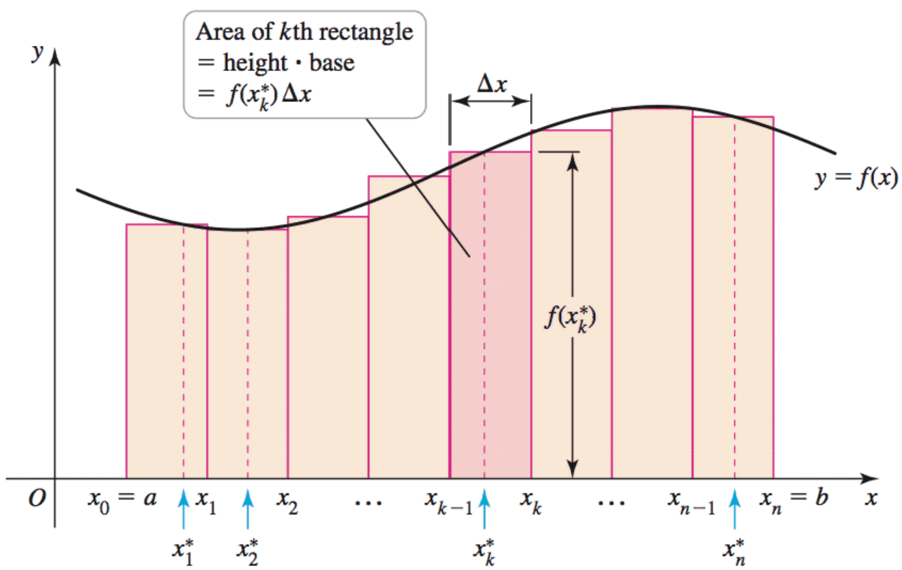
$$\Delta x = \frac{b-a}{n}.$$

Then we subdivide the interval into n subintervals by letting

$$x_0 = a \quad \text{and} \quad x_k = x_{k-1} + \Delta x.$$

Then $x_n = b$. The endpoints $x_0, x_1, \ldots, x_n$ of the subintervals are called ***grid points****. The equally spaced numbers x_k form a* ***regular partition*** *of $[a, b]$.*

Let f be a function defined on an interval $[a, b]$ that we have partitioned into n intervals.

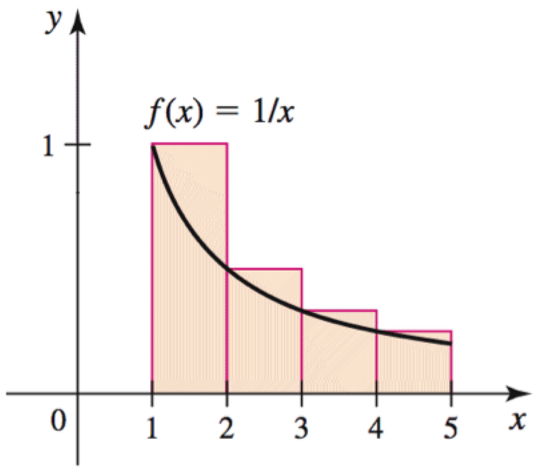


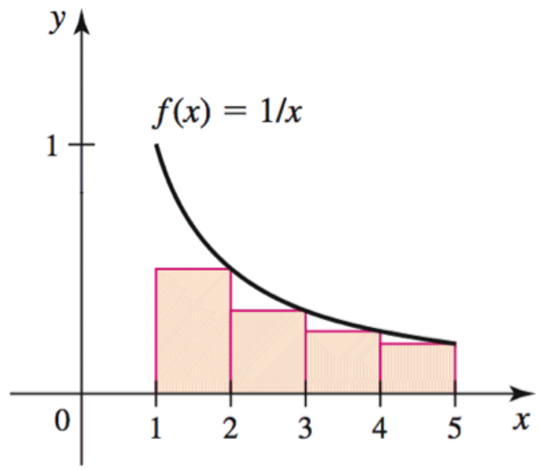
A **Riemann sum** is computed by adding the areas of any rectangles with bases in the subintervals in the partition and height equal to $f(x_k^*)$, where x_k^* is some point in the interval, called a **test point**:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x.$$

- If x_k^* is the left endpoint of $[x_{k-1}, x_k]$ then we call it a **left Riemann sum**
- If x_k^* is the right endpoint of $[x_{k-1}, x_k]$ then we call it a **right Riemann sum**
- If x_k^* is the midpoint of $[x_{k-1}, x_k]$ then we call it a **midpoint Riemann sum**

Example 3 (§5.1, Ex. 24). Calculate the left and right Riemann sums for $f(x) = 1/x$ on $[1, 5]$, with $n = 4$.





Solution. We have $\Delta x = (5 - 1)/4 = 1$, so the subintervals are $[1, 2]$, $[2, 3]$, $[3, 4]$, and $[4, 5]$. The left Riemann sum is

$$f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \approx 2.0833.$$

The right Riemann sum is

$$f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + f(5)\Delta x = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \approx 1.2833.$$

Example 4 (§5.1, Ex. 42). Let $f(x) = 4 - x$ on $[-1, 4]$, and $n = 5$.

(a) Sketch the graph of the function on the given interval.

(b) Calculate Δx and the grid points $x_0, \ldots, x_n$.

(c) Illustrate the midpoint Riemann sum by sketching the appropriate rectangles.

(d) Calculate the midpoint Riemann sum.

1.4 Area under the velocity curve

When we approximate areas under curves using Riemann sums, we can incrementally subdivide the interval into smaller and smaller pieces. This is a very important idea, and our first result about it is the following:

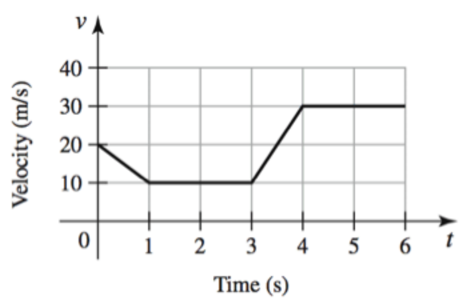
Theorem 5. *If f is a positive continuous function on $[a, b]$ then if we take smaller and smaller partitions of $[a, b]$, the Riemann sums are converging to a number that is the area under the curve between $x = a$ and $x = b$.*

Going back to our original example, when we approximate the displacement by the approximate area under the velocity graph, if we take smaller and smaller rectangles we get better and better approximations. By the theorem above, if we take the limit of this process, our sums converge to the area under the graph that would be the precise displacement for the relevant interval of time. Hence

$$\text{If the velocity is positive, the area under velocity graph between } t_0 \text{ and } t_1$$
$$=$$
$$\text{the displacement between } t_0 \text{ and } t_1.$$

Let's see this in an illustration:

Example 6 (§5.1, Ex. 70)**.** Consider the velocity of an object moving along a line:



(a) Describe the motion of the particle over the interval $[0, 6]$.

(b) Use geometry to find the displacement of the object between $t = 0$ and $t = 3$.

(c) Use geometry to find the displacement of the object between $t = 3$ and $t = 5$.

(d) Assuming that the velocity remains 30 m/s for $t \geq 4$, find the function that gives the displacement between $t = 0$ and any $t \geq 5$.