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Today's topics

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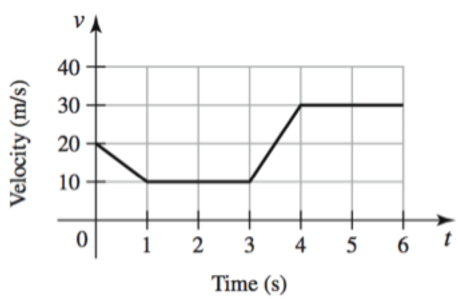
1 Approximating area under curves and Riemann sums

Briggs–Cochran–Gillett §5.1, pp. 338–352

1.1 Area under the velocity curve

Example 1 (§5.1, Ex. 70). Consider the velocity of an object moving along a line:



(a) Describe the motion of the particle over the interval $[0, 6]$.

(b) Use geometry to find the displacement of the object between $t = 0$ and $t = 3$.

(c) Use geometry to find the displacement of the object between $t = 3$ and $t = 5$.

(d) Assuming that the velocity remains 30 m/s for $t \geq 4$, find the function that gives the displacement between $t = 0$ and any $t \geq 5$.

1.2 Sigma notation

When working with Riemann sums, *sigma notation* can be used to express these sums in a compact way.

For example, the sum

$$1 + 2 + 3 + \cdots + 10$$

is written in sigma notation as

$$\sum_{k=1}^{10} k.$$

Here are two useful properties of sigma notation:

- Constant multiple rule: Let c be a constant. Then $\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k$.
- Addition rule: $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.

1.3 Sums of powers

The following formulas for sums of powers of integers are also very useful:

Theorem 2 (Sums of powers of integers)**.** *Let n be a positive integer and c a real number.*

1. $\sum_{k=1}^{n} c = cn$
2. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
3. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Example 3 (§5.1, Ex. 49)**.** Evaluate the following expressions:

1. $\sum_{k=1}^{10} k$
2. $\sum_{k=1}^{6} (2k+1)$
3. $\sum_{p=1}^{5} (2p + p^2)$
4. $\sum_{n=0}^{4} \sin \frac{n\pi}{2}$

1.4 Riemann sums using sigma notation

With sigma notation, a Riemann sum has the convenient compact form

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

We can use this to rewrite left, right, and midpoint Riemann sums:

Definition 4 (Left, right, and midpoint Riemann sums in sigma notation)**.** *Suppose f is defined on an interval $[a,b]$, which is divided into n subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$, then the Riemann sum for f on $[a,b]$ is $\sum_{k=1}^{n} f(x_k^*)\Delta x$. Here are our three cases:*

1. *Left Riemann sum: $x_k^* = a + (k-1)\Delta x$*
2. *Right Riemann sum: $x_k^* = a + k\Delta x$*
3. *Midpoint Riemann sum: $x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$*

Example 5 (§5.1, Ex. 52)**.** Let $f(x) = x^2 + 1$ on $[-1, 1]$. Letting $n = 50$, use sigma notation to write the left, right, and midpoint Riemann sums.

2 Definite integrals

Briggs–Cochran–Gillett §5.2 pp. 353–367

2.1 Net area

So far, we have been considering functions f which are nonnegative on an interval $[a,b]$. Now we will discover the geometric meaning of Riemann sums when f is negative on some or all of $[a,b]$.

Consider the function $f(x) = 1 - x^2$ on the interval $[1,3]$ with $n = 4$. We compute a midpoint Riemann sum. The length of each subinterval is $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$. The grid points are

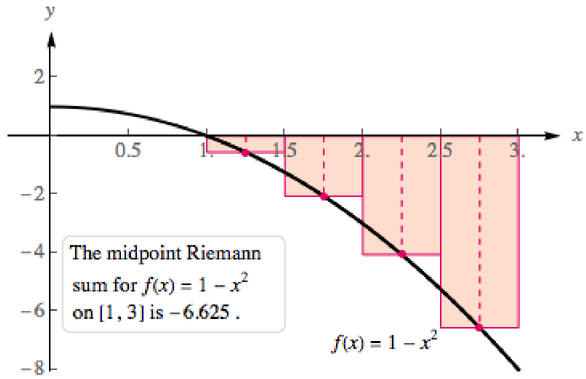
$$x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3.$$

We compute the midpoints of the subintervals:

$$x_1^* = 1.25, x_2^* = 1.75, x_3^* = 2.25, x_4^* = 2.75.$$

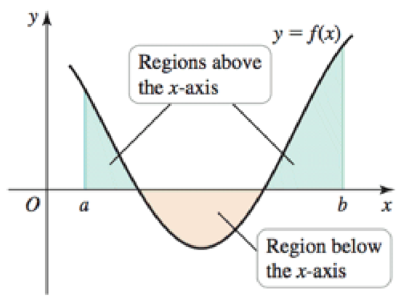
So the midpoint Riemann sum is

$$\sum_{k=1}^{4} f(x_k^*)(0.5) = (f(1.25) + f(1.75) + f(2.25) + f(2.75))(0.5) = -6.625.$$



Note that all values of $f(x_k^*)$ are negative, so the Riemann sum is also negative. Indeed, the Riemann sum is an approximation to the *negative* of the area of the region bounded by the curve and the x-axis.

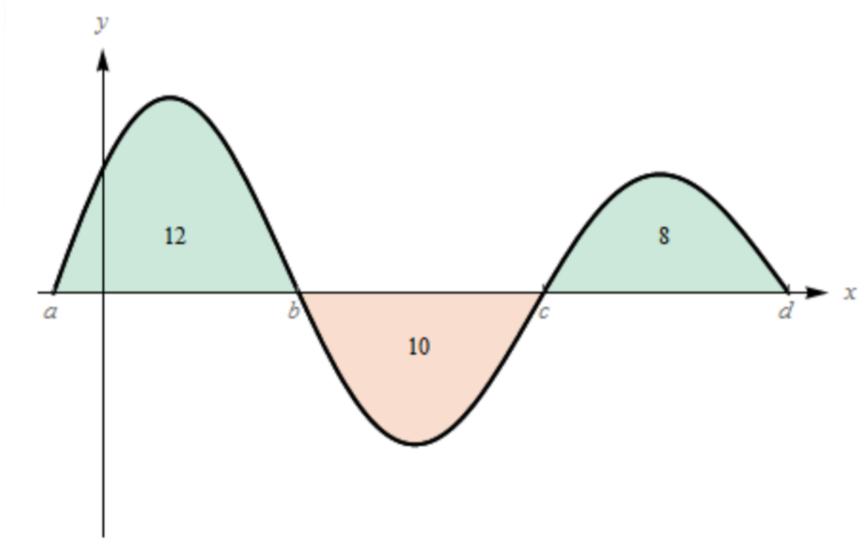
More generally, if f is positive on only part of $[a, b]$, we get positive contributions to the Riemann sum where f is positive and negative contributions to the Riemann sum where f is negative. In this case, Riemann sums approximate the area of the regions that lie above the x-axis minus the area of the regions that lie below the x-axis. This difference between the positive and negative contributions is called the ***net area***. It can be positive, negative, or zero.



Definition 6 (Net area)**.** *Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The net area of R is the sum of the areas of the parts of R that lie above the x-axis minus the area of the parts of R that lie below the x-axis on $[a, b]$.*

Geometrically:

The *definite integral* corresponds to the net area:



We have $\int_a^b f(x)dx = 12$, $\int_b^c f(x)dx = -10$, $\int_a^d f(x)dx = 10$.

Below is a formal definition.

2.2 Definition of definite integral

Riemann sums for f on $[a,b]$ approximate the net area of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$. How do we make these approximations exact? If f is continuous on $[a,b]$, it is reasonable to expect the Riemann sum approximations to approach the exact value of the net area as the number of subintervals $n \to \infty$ and as the length of the subintervals $\Delta x \to 0$, giving net area $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$. This brings us to the notion of the definite integral:

Definition 7 (Definite integral)**.** *A function f defined on $[a,b]$ is integrable on $[a,b]$ if the limit $\lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists. This limit is the **definite integral of** f **from** a **to** b, which we write*

$$\int_a^b f(x)dx = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

Example 8. Let us compute the definite integral $\int_0^1 x^2\,dx$ using a right Riemann sum. We will use a regular partition, so $\Delta x_k = 1/n$ and $x_k = k/n$. Since we are taking a right Riemann sum, $x_k^* = x_k = k/n$. So

$$\begin{aligned}\int_0^1 x^2\,dx &= \lim_{\Delta\to 0} \sum_{k=1}^{n} (x_k^*)^2 \Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_k^2 \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} \\ &= \lim_{n\to\infty} \frac{1}{n^3} \cdot \sum_{k=1}^{n} k^2 = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} \\ &= \frac{2}{6} = \frac{1}{3}.\end{aligned}$$