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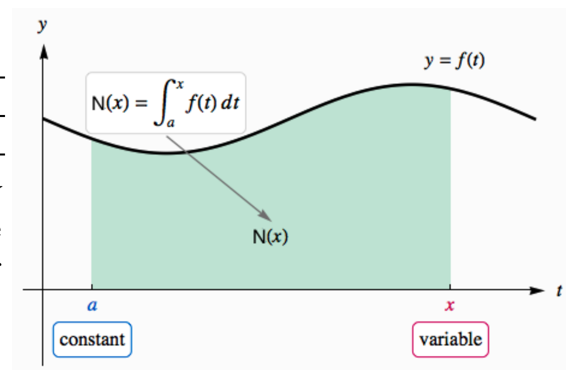
Today's topics

1 Net area functions and the Fundamental Theorem of Calculus

Briggs–Cochran–Gillett §5.3, pp. 367–381

1.1 Net area functions

The concept of net area is essential in understanding the relationship between derivatives and integrals. If instead of finding the net area of a continuous function over a fixed interval $[a, b]$, we allow the right boundary point to move and calculate the net area over the intervals $[a, x]$, the net area for the different values of f define a function:



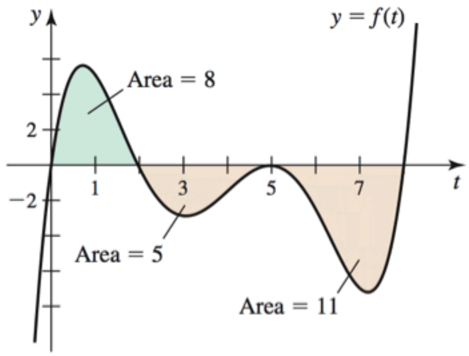
Definition 1. *Let f be a continuous function, for $t \geq a$. The* ***net area function for* f *with left endpoint* a** *is*

$$N(x) = \int_a^x f(t)dt,$$

where $x \geq a$.

Remark 2. In the textbook, "net area" functions are called "area functions." We call them **net area** functions to make it clear that they calculate a *net* area and not an area!

Example 3 (§5.3, Ex. 14).



The graph of f is shown in the figure. Let $N(x) = \int_0^x f(t)dt$ and $F(x) = \int_2^x f(t)dt$ be two net area functions for f. Evaluate

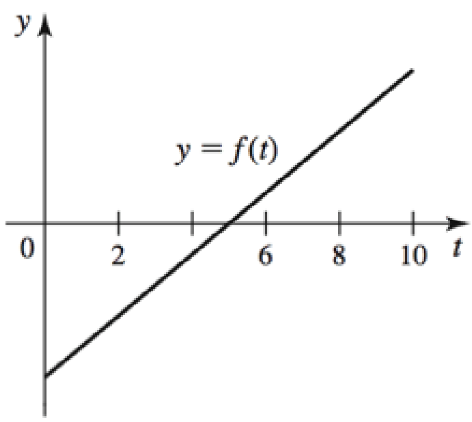
1. $N(2)$ 3. $N(0)$ 5. $N(8)$ 7. $F(2)$
2. $F(5)$ 4. $F(8)$ 6. $N(5)$

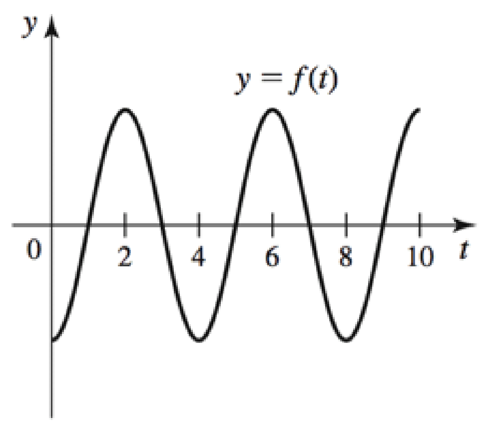
Example 4 (§5.3 Ex. 88, 90). For each of the functions below, consider the function f given by its graph.

(a) Estimate the zeros of the net area function $N(x) = \int_0^x f(t)dt$ for $0 \leq x \leq 10$.

(b) Estimate the points (if any) at which N has a local maximum or minimum.

(c) Sketch a graph of N for $0 \leq x \leq 10$ (without a scale on the y-axis).

1. 

2. 

1.2 The Fundamental Theorem of Calculus (FTC)

Let's look at the following example of different net area functions for a linear function:

Example 5 (§5.3, Ex. 17)**.** Let $f(t) = t$ and consider the two net area functions $N(x) = \int_0^x f(t)dt$ and $F(x) = \int_2^x f(t)dt$.

(a) Evaluate $N(2)$ and $N(4)$. Then use geometry to find an expression for $N(x)$, $x \geq 0$.

(b) Evaluate $F(4)$ and $F(6)$. Then use geometry to find an expression for $F(x)$, $x \geq 0$.

(c) Show that $N(x) - F(x)$ is a constant and that $N'(x) = F'(x) = f(x)$.

This example suggests that the net area function $N(x)$ is an **antiderivative** of f. In fact, this holds for more general functions:

Theorem 6 (Fundamental Theorem of Calculus (FTC))**.** *Let f be a continuous function on $[a, b]$.*

1. *The net area function $N(x) = \int_a^x f(t)dt$ for $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b) and we have*

$$N'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x),$$

which means the net area function of f is an antiderivative of f on $[a, b]$.

2. *If F is any antiderivative of f on $[a, b]$ then*

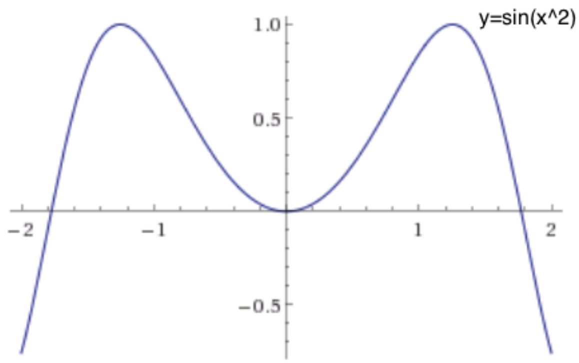
$$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b.$$

Part 2 of the FTC is a powerful method for evaluating definite integrals and it is a direct consequence of part 1 (see p. 366 in the textbook).

Hence we have a new method to compute definite integrals $\int_a^b f(x)dx$:

- find any antiderivative of f; call it F;
- compute $F(b) - F(a)$.

Remark 7. This method only works when we can find a formula for an antiderivative for f. This is only the case for a relatively small group of functions! The definition of integral is still the limit of Riemann sums and geometrically is given by the net area between the graph and the x-axis in the given interval. You should always remember this! (For example, functions like $\sin(x^2)$ and e^{x^2} do not have antiderivatives given by closed-form formulas, but they are continuous and hence integrable in any closed interval.)



Example 8. Simplify the following expressions.

a) $\dfrac{d}{dx}\displaystyle\int_{x^2}^{10} \frac{1}{z^2+1}dz$ b) $\dfrac{d}{dx}\displaystyle\int_{-x}^{x} \sqrt{1+t^2}dt$ c) $\dfrac{d}{dx}\displaystyle\int_{e^x}^{e^{2x}} \ln(t^2)dt$

Example 9. Evaluate the following integrals using the Fundamental Theorem of Calculus.

1. $\int_0^2 (3x^2 + 2x)dx$

2. $\int_4^9 \frac{2+\sqrt{t}}{t} dt$

3. $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

4. $\int_0^{\pi} (1 - \sin x)dx$

5. $\int_{\pi/16}^{\pi/8} 8 \csc^2 4x dx$

6. $\int_0^{\pi/8} \cos 2x dx$