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Today's topics

1 The Fundamental Theorem of Calculus 1

1 The Fundamental Theorem of Calculus

Briggs–Cochran–Gillett §5.3, pp. 367–381

Recall the notion of a net area function:

Definition 1. *Let f be a continuous function, for $t \geq a$. The* ***net area function for f with left endpoint a*** *is*

$$N(x) = \int_a^x f(t)dt,$$

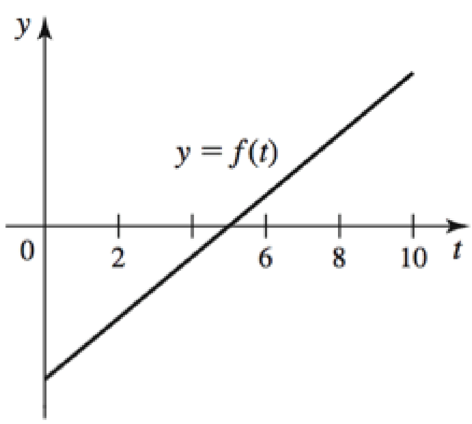
where $x \geq a$.

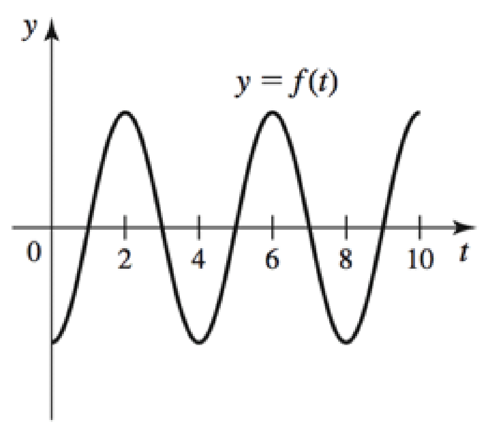
Example 2. For each of the functions below, consider the function f given by its graph.

(a) Estimate the zeros of the net area function $N(x) = \int_0^x f(t)dt$ for $0 \leq x \leq 10$.

(b) Estimate the points (if any) at which N has a local maximum or minimum.

(c) Sketch a graph of N for $0 \leq x \leq 10$ (without a scale on the y-axis).

1. 

2. 

Theorem 3 (Fundamental Theorem of Calculus (FTC)). *Let f be a continuous function on $[a, b]$.*

1. *The net area function $N(x) = \int_a^x f(t)dt$ for $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b) and we have*

$$N'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x),$$

which means the net area function of f is an antiderivative of f on $[a, b]$.

2. *If F is any antiderivative of f on $[a, b]$ then*

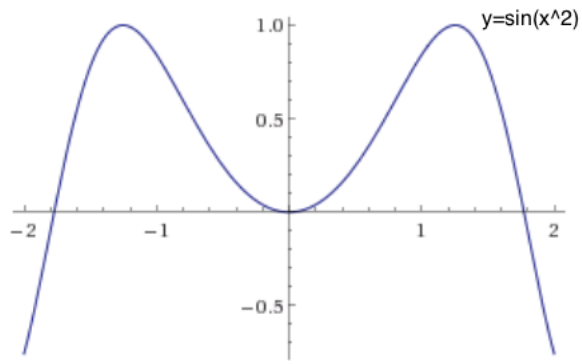
$$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b.$$

Part 2 of the FTC is a powerful method for evaluating definite integrals and it is a direct consequence of part 1 (see p. 366 in the textbook).

Hence we have a new method to compute definite integrals $\int_a^b f(x)dx$:

- find any antiderivative of f; call it F;
- compute $F(b) - F(a)$.

Remark 4. This method only works when we can find a formula for an antiderivative for f. This is only the case for a relatively small group of functions! The definition of integral is still the limit of Riemann sums and geometrically is given by the net area between the graph and the x-axis in the given interval. You should always remember this! (For example, functions like $\sin(x^2)$ and e^{x^2} do not have antiderivatives given by closed-form formulas, but they are continuous and hence integrable in any closed interval.)



Example 5. Simplify the following expressions.

(a) $\dfrac{d}{dx}\displaystyle\int_{x^2}^{10} \frac{1}{z^2+1}\,dz$

(b) $\dfrac{d}{dx}\displaystyle\int_{-x}^{x} \sqrt{1+t^2}\,dt$

(c) $\dfrac{d}{dx}\displaystyle\int_{e^x}^{e^{2x}} \ln(t^2)\,dt$

Solution. (a) Let F be any antiderivative of $\dfrac{1}{z^2+1}$. By FTC,

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{10} \frac{1}{z^2+1}\,dz &= \frac{d}{dx}\Big(F(10) - F(x^2)\Big) = \frac{d}{dx}F(10) - \frac{d}{dx}F(x^2)\\ &= 0 - F'(x^2)\cdot 2x = -\frac{1}{(x^2)^2+1}\cdot 2x = \frac{-2x}{x^4+1}.\end{aligned}$$

(b) Let F be any antiderivative of $\sqrt{1+t^2}$. By FTC,

$$\begin{aligned}\frac{d}{dx}\int_{-x}^{x} \sqrt{1+t^2}\,dt &= \frac{d}{dx}\left(F(x) - F(-x)\right) = F'(x) - F'(-x)\cdot(-1)\\ &= \sqrt{1+x^2} + \sqrt{1+(-x)^2} = 2\sqrt{1+x^2}.\end{aligned}$$

(c) Let F be any antiderivative of $\ln(t^2)$. By FTC,

$$\begin{aligned}\frac{d}{dx}\int_{e^x}^{e^{2x}} \ln(t^2)\,dt &= \frac{d}{dx}\Big(F(e^{2x}) - F(e^x)\Big) = F'(e^{2x})\cdot 2e^{2x} - F'(e^x)\cdot e^x\\ &= 2e^{2x}\ln((e^{2x})^2) - e^x\ln((e^x)^2) = 8xe^{2x} - 2xe^x.\end{aligned}$$

Example 6. Evaluate the following integrals using the Fundamental Theorem of Calculus.

(a) $\displaystyle\int_4^9 \frac{2+\sqrt{t}}{t}dt = \int_4^9 (2t^{-1}+t^{-1/2})\,dt = \left(2\ln|t| + 2t^{1/2}\right)\Big|_4^9 = 2\ln(9) + 2\sqrt{9} - 2\ln(4) - 2\sqrt{4}$

(b) $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)\Big|_0^{1/2} = \sin^{-1}(1/2) - \sin^{-1}(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

(c) $\displaystyle\int_0^{\pi} (1-\sin x)dx = x + \cos x|_0^{\pi} = \pi + \cos\pi - 0 - \cos 0 = \pi - 2$

(d) $\displaystyle\int_{\pi/16}^{\pi/8} 8\csc^2(4x)dx = -2\cot(4x)|_{\pi/16}^{\pi/8} = -2\cot\frac{\pi}{2} + 2\cot\frac{\pi}{4} = 0 + 2 = 2$

(e) $\displaystyle\int_0^{\pi/8} \cos(2x)dx = \frac{1}{2}\sin(2x)\Big|_0^{\pi/8} = \frac{1}{2}\sin\frac{\pi}{4} - \frac{1}{2}\sin 0 = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{4}$