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Today's topics

1 Midterm 2 review

This class was entirely dedicated to midterm 2 review. There is one problem that was asked in class that I deferred because it was a fair bit more difficult than the problems on that topic that might appear on the exam. Here's a solution to that problem, in case you're curious.

Example 1 (§4.5, Ex. 63: Watching a Ferris wheel)**.** An observer stands 20 m from the bottom of a Ferris wheel on a line that is perpendicular to the face of the wheel, with their eyes at the level of the bottom of the wheel. The wheel revolves at a rate of π rad/min, and the observer's line of sight to a specific seat on the Ferris wheel makes an angle θ with the horizontal. At what time during a full revolution is θ changing most rapidly?

Solution. Let r be the radius of the Ferris wheel, and let α be the angle in $[0, 2\pi)$ that the specific seat on the Ferris wheel makes with the center of the wheel, where we follow the convention that $\alpha = 0$ when the seat is at the rightmost edge of the Ferris wheel from the perspective of the observer.

In the plane of the Ferris wheel, setting the origin to be the base of the Ferris wheel, the specific seat on the Ferris wheel thus has coordinates $(r\cos\alpha, r + r\sin\alpha)$. So the distance from the base of the Ferris wheel to the seat is

$$\begin{aligned} d = \sqrt{(r\cos\alpha)^2 + (r + r\sin\alpha)^2} &= r\sqrt{1 - \sin^2\alpha + 1 + 2\sin\alpha + \sin^2\alpha} \\ &= r\sqrt{2 + 2\sin\alpha} = r\sqrt{2}\sqrt{1 + \sin\alpha}. \end{aligned}$$

Consider the right triangle whose vertices are the observer, the base of the Ferris wheel, and the seat. One leg has length 20 m, the other length we have just computed above, and the angle at the observer is θ. Thus,

$$\tan\theta = \frac{r\sqrt{2}}{20}\sqrt{1 + \sin\alpha}.$$

Both θ and α are functions of the time t (measured in minutes), so we differentiate both sides with respect to t:

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{r\sqrt{2}}{20} \cdot \frac{\cos\alpha}{2\sqrt{1 + \sin\alpha}} \cdot \frac{d\alpha}{dt} = \frac{\pi r\sqrt{2}}{20} \cdot \frac{\cos\alpha}{2\sqrt{1 + \sin\alpha}},$$

where the second equality is because $d\alpha/dt = \pi$ rad/min by assumption. Thus,

$$\begin{aligned}\left|\frac{d\theta}{dt}\right| &= \frac{\pi r\sqrt{2}}{40}\cdot\frac{\cos^2(\theta)|\cos\alpha|}{\sqrt{1+\sin\alpha}}\\ &= \frac{\pi r\sqrt{2}}{40}\cdot\frac{\cos^2(\theta)|\cos\alpha|}{\sqrt{1+\sin\alpha}}\frac{\sqrt{1-\sin\alpha}}{\sqrt{1-\sin\alpha}}\\ &= \frac{\pi r\sqrt{2}}{40}\cdot\frac{\cos^2(\theta)|\cos\alpha|\sqrt{1-\sin\alpha}}{\sqrt{1-\sin^2\alpha}}\\ &= \frac{\pi r\sqrt{2}}{40}\cos^2(\theta)\sqrt{1-\sin\alpha}.\end{aligned}$$

We want to find where this quantity is as large as possible. The largest $\cos^2(\theta)$ can be is 1, and the largest $\sqrt{1-\sin\alpha}$ can be is $\sqrt{2}$. These both occur when the seat is at the bottom of the Ferris wheel, and in fact this is the *unique* maximum because $\sqrt{1-\sin\alpha} < \sqrt{2}$ for any value of α in $[0, 2\pi)$ other than $\frac{3\pi}{2}$ (corresponding to the bottom of the wheel).