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Today's topics

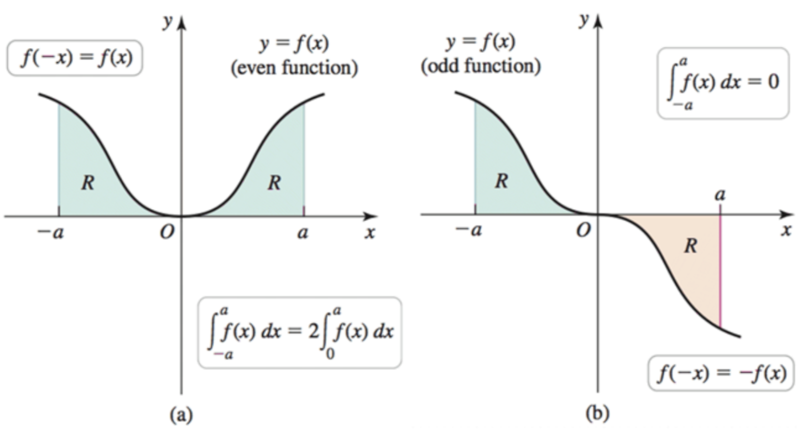
1 Working with integrals

Briggs–Cochran–Gillett §5.4, pp. 381–387

1.1 Even and odd functions

Definition 1 (Even and odd functions).

1. *An* ***even*** *function f is a function that satisfies $f(-x) = f(x)$. This means its graph is symmetric about the y-axis. Ex:* $\cos(x)$, x^2, x^4.
2. *An* ***odd*** *function f is a function that satisfies $f(-x) = -f(x)$. This means its graph is symmetric about the origin. Ex:* $\sin(x)$, x, x^3.



Theorem 2. *Let a be a positive number and let f be an integrable function on the interval $[-a, a]$. Then*

1. *if f is even,* $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$;
2. *if f is odd,* $\int_{-a}^{a} f(x)dx = 0$.

Example 3 (§5.4, Ex. 20)**.** Use symmetry to evaluate the following integral

$$\int_{-1}^{1}(1-|x|)dx.$$

Solution. Observe that

$$1-|-x|=1-|x|,$$

so the integrand is an even function. Thus, by the above theorem,

$$\int_{-1}^{1}(1-|x|)dx=2\int_{0}^{1}(1-|x|)dx=2\int_{0}^{1}(1-x)dx=2\left(x-\frac{1}{2}x^2\right)\Big|_{0}^{1}=2\left(1-\frac{1}{2}-(0-0)\right)=1.$$

Example 4 (§5.4, Ex. 4)**.** Suppose f is an odd function, $\int_0^4 f(x)dx=3$, and $\int_0^8 f(x)dx=9$. Evaluate

1. $\int_{-4}^{8} f(x)dx$
2. $\int_{-8}^{4} f(x)dx$

Solution. Using properties of integration and the above theorem, we have

$$\int_{-4}^{8} f(x)dx=\int_{-4}^{0} f(x)dx+\int_{0}^{8} f(x)dx=-\int_{0}^{4} f(x)dx+\int_{0}^{8} f(x)dx=-3+9=6.$$

For the second part, we can use this computation and the fact that f is odd to compute

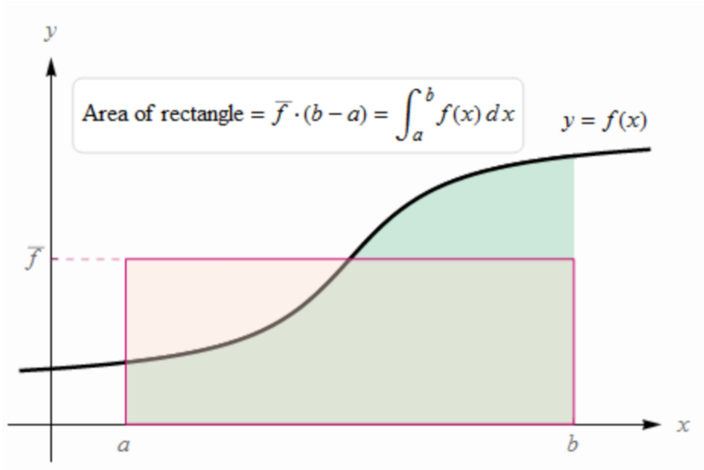
$$\int_{-8}^{4} f(x)dx=-\int_{-4}^{8} f(x)dx=-6.$$

(Alternatively, we could do the second computation using the same method as the first.)

1.2 Average value of a function

Definition 5 (Average value of a function)**.** *The average value of an integrable function on the interval* $[a, b]$ *is*

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$



Example 6 (§5.4, Ex. 29)**.** Find the average value of $f(x) = \cos x$ on $[-\pi/2, \pi/2]$. Draw a graph of the function and indicate the average value.

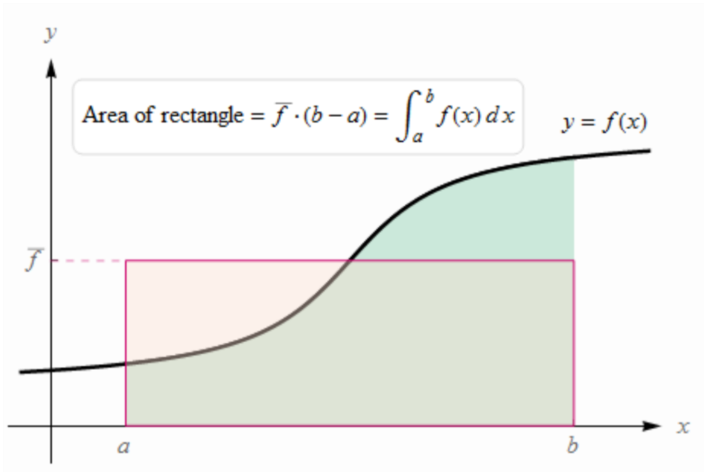
Solution. The average value of the function is

$$\begin{aligned}
\frac{1}{\pi/2 - (-\pi/2)}\int_{-\pi/2}^{\pi/2} \cos(x)dx &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(x)dx \\
&= \frac{1}{\pi}\sin(x)\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{1}{\pi}\left(\sin(\pi/2) - \sin(-\pi/2)\right) \\
&= \frac{1}{\pi}(1 - (-1)) \\
&= \frac{2}{\pi}.
\end{aligned}$$

1.3 Mean Value Theorem for integrals

Theorem 7. *Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that*

$$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$



Example 8 (§5.4, Ex. 43)**.** Find or approximate all points at which the function $f(x) = 1-|x|$ equals its average value on the interval $[-1, 1]$.

Solution. The average value of the function f on the interval $[-1, 1]$ is

$$\frac{1}{1-(-1)}\int_{-1}^{1}(1-|x|)dx = \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

(We computed this integral earlier today.)

So we want to find solutions with $-1 < c < 1$ to the equation

$$\frac{1}{2} = f(c) = 1 - |c|.$$

Thus we obtain $|c| = 1/2$, so there are two solutions: $c = 1/2$ and $c = -1/2$.